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THÈSE

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CONTRIBUTIONS TO THE STUDY OF LÉVY PROCESSES AND FRACTIONAL PROCESSES VIA THE MALLIAVIN CALCULUS AND APPLICATIONS IN STATISTICS.

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Résumé

Cette thèse se décompose en six chapitres plus ou moins distincts. Cependant, tous font appel au calcul de Malliavin, aux notions de processus gaussien et processus de Lévy, et à leur utilisation en statistique. Chacune des trois parties a fait l'objet de deux articles.
Dans la première partie, nous établissons les théorèmes d'Itô et de Tanaka pour le mouvement brownien bifractionnaire multidimensionnel. Ensuite nous étudions l'existence de la densité d'occupation pour certains processus en relation avec le mouvement brownien fractionnaire.
Dans la deuxième partie, nous analysons, dans un premier temps, le comportement asymptotique de la variation cubique pour le processus de Rosenblatt. Dans un deuxième temps, nous construisons d'une part des estimateurs efficace pour la dérive de mouvement brownien fractionnaire et d'autre part des estimateurs biaisés de type James-Stein qui dominent, sous le riqsue quadratique usuel, l'estimateur du maximum de vraisemblance.
La dernière partie présente deux travaux. Dans le premier, nous utilisons une approche menant à un calcul de Malliavin pour les processus de Lévy, qui a été développée récemment par Solé et al. [106], et nous étudions des processus anticipés de type intégrale d'Itô-Skorohod (au sens de [111]) sur l'espace de Lévy. Dans le deuxième, nous étudions le lien entre les processus stables et les processus auto-similaires, à travers des processus qui sont infiniment divisibles en temps.

Abstract

In the first part, we establish Itô's and Tanaka's formulas for the multidimensional bifractional Brownian motion. We study the existence of an occupation density for certain processes related to fractional Brownian motion.
In the second part, we study the cubic variation of Rosenblatt process. We consider the problem of efficient estimation for the drift of fractional Brownian motion . We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.
In the last part, we study Skorohod integral processes on Lévy spaces and we prove an equivalence between this class of processes and the class of Itô-Skorohod process. We give a link between stable proceses and selfsimilaire processes through stochastic processes which are infinitely divisible with respect to time .

Contents

Introduction et principaux résultats

Au cours de cette introduction, nous présenterons les thèmes étudiés et mettrons en valeur les résultats principaux.

L'objet de cette thèse est une contribution, via le calcul stochastique (le calcul de Malliavin en premier lieu), à l'étude de certains processus stochastiques, gaussiens ou non-gaussiens, liés au mouvement brownien fractionnaire et aux processus de Lévy. Nous avons divisé le présent manuscrit en trois parties, chacune ayant deux chapitres. D'abord, nous étudions une classe de processus gaussiens, ayant la propriété de quasi-hélice au sens de Kahane ([58], [59]) et qui ne sont pas nécessairement des processus de Volterra, en particulier le mouvement brownien bifractionnaire (mBbif). Deuxièmement, nous nous intéressons à l'analyse du comportement asymptotique de la variation cubique pour le processus de Rosenblatt et, troisièmement, à l'étude des processus de type Itô-Skorohod sur un espace de Lévy. Enfin, nous étudions, à l'aide de processus qui sont infiniment divisibles en temps, le lien entre les processus stables et les processus auto-similaires.

Les diverses approches d'analyse stochastique, pour étudier des processus qui ne sont pas forcément des semimartingales, peuvent être divisées en deux principales catégories : celles qui reposent sur les propriétés trajectorielles (par exemple, la théorie des trajectoires rugueuses [100] et le calcul stochastique par régularisation [102]) et celles qui utilisent le calcul de Malliavin via le caractère gaussien. Dans cette thèse, nous nous intéressons principalement à la deuxième approche.

Le calcul de Malliavin, aussi connu sous le nom de "calcul stochastique des variations", est un calcul différentiel sur un espace de dimension infinie, l'espace de Wiener $\mathcal{C}\left([0,1],\mathbb{R}^d\right)$. Introduit par Paul Malliavin [67] en 1976, il fut conçu à l'origine pour étudier la régularité des densités des solutions des équations différentielles stochastiques. Le premier fait marquant de ce calcul est qu'il a permis de fournir une preuve probabiliste du célèbre théorème d'Hörmander, donnant une condition d'hypoellipticité pour les opérateurs aux dérivées partielles. Dans les années qui ont suivi, de nombreux probabilistes ont travaillé sur ce sujet et la théorie a été principalement développée soit dans le cadre de l'analyse sur l'espace de Wiener, soit dans le cadre du bruit blanc. Plusieurs applications en calcul stochastique sont apparues. Il existe plusieurs ouvrages de référence sur le sujet, parmi lesquels aux de D. Nualart [81], D. Bell [9], D. Ocone [88], B. Øksendal [90].

L'application du calcul de Malliavin en finance date du début des années 90. En 1991 Karatzas et Ocone montrèrent comment le calcul de Malliavin peut être utilisé pour calculer des portefeuilles de couverture en marché complet [89]. Depuis, le calcul de Malliavin a suscité un intérêt croissant et beaucoup d'autres applications en finance ont été révélées.

Plus récemment, le calcul de Malliavin est aussi devenu un outil puissant pour développer le calcul stochastique pour des processus gaussiens qui ne sont pas nécessairement des semimartingales (voir, par exemple, Watanabe [116], Nualart [81], Malliavin [68] et Kruk et al. [61]), en particulier pour les processus de Volterra, généralisation du mouvement brownien fractionnaire, et définis par (voir, [28]):

$$\left\{ B_t = \int_0^t K(t,s)dW_s, \quad t \in [0,T] \right\} \tag{1}$$

où K est un noyau déterministe et W un mouvement brownien standard. Les processus de Volterra, avec leurs nombreuses propriétés intéressantes, constituent un domaine d'étude en plein développement actuellement.

Comme le calcul de Malliavin joue un rôle important dans nos travaux, nous allons maintenant en rappeler les principaux résultats.

Calcul de Malliavin sur un espace gaussien

Considérons un espace de probabilité $(\Omega, \mathcal{F}, P)$ sur lequel est défini un processus gaussien centré $(B_t, t \in [0,T])$, et où $\mathcal{F}$ est la tribu engendrée par B.
Notons $\mathcal{E}$ la classe des fonctions simples sur l'intervalle $[0,T]$. Soit $\mathcal{H}$ l'espace de Hilbert défini comme la fermeture de $\mathcal{E}$ relativement au produit scalaire

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}} = R(t,s).$$

L'application $1_{[0,t]} \longrightarrow B_t$ peut-être étendue en une isométrie entre $\mathcal{H}$ et l'espace gaussien associé au processus B. Introduisons maintenant des propriétés fondamentales du calcul de Malliavin relativement au processus gaussien B, qui seront utilisées de manière récurrente au cours de ce travail.

Opérateur de dérivation. Notons $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ la classe des fonctions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ infiniment dérivables et telles que f et toutes ses dérivées partielles sont bornées. Notons $\mathcal{S}$ la classe des fonctionnelles cylindriques de la forme

$$F = f\left(B(\varphi_1), \ldots, B(\varphi_n)\right) \tag{2}$$

où $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ et $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$.
L'opérateur de dérivation, au sens de Malliavin, d'une fonctionnelle F de la forme (2) est alors l'application $D : \mathcal{S} \longrightarrow L^2(\Omega; \mathcal{H})$ définie par

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(B(\varphi_1), \ldots, B(\varphi_n)\right) \varphi_i.$$

Soient $k \geq 1$ et $p \geq 1$. On note $D^{k,p}$ l'espace de Sobolev défini comme la fermeture de $\mathcal{S}$ par rapport à la norme

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^{k} \|D^j F\|_{L^p(\Omega;\mathcal{H}^{\otimes j})}^p,$$

où D^j est l'opérateur de dérivation itérée de D.

Opérateur de Skorohod. L'opérateur de Skorohod,(ou opérateur de divergence), noté δ, est l'adjoint de l'opérateur D, défini grâce à la dualité

$$E(F\delta(u)) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right); \quad u \in L^2(\Omega;\mathcal{H}), F \in D^{1,2}. \tag{3}$$

Le domaine de δ, noté $Dom(\delta)$, est l'ensemble des $u \in L^2\left(\Omega;\mathcal{H}\right)$ tel que

$$E(\langle DF, u\rangle_{\mathcal{H}}) \leq C\sqrt{E(F^2)},$$

pour tout $F \in D^{1,2}$, où C est une constante pouvant dépendre de u. Par la suite, l'intégrale de Skorohod $\delta(u)$ sera aussi notée par

$$\delta(u) = \int_0^T u_s \delta B_s.$$

Formule d'intégration par parties. Soient $F \in D^{1,2}$ et $u \in \mathrm{Dom}(\delta)$. Supposons que les éléments aléatoires Fu et $F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}$ sont de carré integrable. Alors

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}.$$

Soit (u_n) une suite de $Dom(\delta)$, qui converge vers u dans $L^2(\Omega;\mathcal{H})$. Supposons que $\delta(u_n)$ converge dans $L^1(\Omega)$ vers une variable aléatoire de carré integrable G. Alors, on obtient

$$u \in Dom(\delta) \text{ et } \delta(u) = G.$$

Formule de commutation. Soient $F \in D^{1,2}$ et $u \in Dom(\delta)$ tels que $F\delta(u)$ est de carré intégrable. Alors $Fu \in Dom(\delta)$ et

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}.$$

Intégrales stochastiques multiples. Supposons maintenant que B est un mouvement brownien, alors $\mathcal{H} = L^2([0,T])$. On introduit l'ensemble $\mathcal{S}_n$ sur lequel on va définir l'intégrale multiple par rapport au mouvement brownien B.

$\mathcal{S}_n$ désigne l'ensemble des fonctions simples à n variables de la forme

$$f = \sum_{i_1,\ldots,i_m=1}^{n} c_{i_1\ldots i_m} 1_{A_{i_1}\times\ldots\times A_{i_m}}$$

où $c_{i_1\ldots i_m} = 0$ si deux indices i_k et i_l sont égaux et où les ensembles boréliens $A_i \in \mathcal{B}([0,T])$ sont disjoints deux à deux. Pour une telle fonction f on définit

$$I_n(f) = \sum_{i_1,\ldots,i_m=1}^{n} c_{i_1\ldots i_m} B(A_{i_1})\ldots B(A_{i_m})$$

où l'on a noté $B(A) := B(1_A)$ pour $A \in \mathcal{B}([0,T])$. On remarque que pour tout $n \geq 1$, I_n est une application linéaire continue entre $\mathcal{S}_n$ et $L^2(\Omega)$. I_n vérifie : pour chaque $h \in \mathcal{H}$ avec $\|h\|_{\mathcal{H}} = 1$, on a $I_n(h^{\otimes^n}) = n! H_n(B(h))$, où $H_n(x)$ est le nième polynôme d'Hermite donné par

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}) \quad \text{pour tout } n \geq 1,$$

avec $H_0(x) = 1$.
On définit $\mathcal{H}_n$, le nième chaos de Wiener, par la fermeture dans $L^2(\Omega)$ du sous-espace vectoriel engendré par $\{H_n(B(h)); h \in \mathcal{H}, \|h\| = 1\}$.
Ainsi, on obtient

$$E\left[I_n(f)I_m(g)\right] = n! \langle \tilde{f}, \tilde{g} \rangle_{L^2([0,1]^n)} \text{ si } m = n \tag{4}$$

et

$$E\left[I_n(f)I_m(g)\right] = 0 \text{ si } m \neq n.$$

De plus, on a

$$I_n(f) = I_n\left(\tilde{f}\right)$$

où $\tilde{f}$ désigne la fonction symétrique de f définie par $\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Comme l'ensemble $\mathcal{S}_n$ est dense dans $L^2([0,1]^n)$ pour tout $n \geq 1$, l'opérateur I_n peut être étendu à une application linéaire continue de $L^2([0,1]^n)$ à $L^2(\Omega)$ et les propriétés ci-dessus sont toujours vraies pour cette cette extension. Notons que I_n peut aussi s'écrire comme une intégrale stochastique itérée

$$I_n(f) = n! \int_0^1 dB_{t_n} \int_0^{t_n} \ldots \int_0^{t_2} dB_{t_1} f(t_1, \ldots, t_n);$$

ici les intégrales sont au sens d'Itô.

Le produit de deux intégrales multiples peut s'écrire comme une somme finie d'intégrales multiples (voir [81]). Plus précisément, si $f \in L^2([0,1]^n)$ et $g \in L^2([0,1]^m)$ sont des fonctions symétriques, alors

$$I_n(f)I_m(g) = \sum_{l=0}^{m \wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g)$$

où la contraction $f \otimes_l g$, appartenant à $L^2([0,1]^{m+n-2l})$, est donnée par

$$\begin{aligned} & (f \otimes_l g)(s_1, \ldots, s_{n-l}, t_1, \ldots, t_{m-l}) \\ = & \int_{[0,1]^l} f(s_1, \ldots, s_{n-l}, u_1, \ldots, u_l) g(t_1, \ldots, t_{m-l}, u_1, \ldots, u_l) du_1 \ldots du_l. \end{aligned}$$

Maintenant, nous allons donner une vue d'ensemble des propriétés probabilistes du mouvement brownien fractionnaire. Nous l'utiliserons comme processus de référence dans les chapitres 1, 2 et 4.

Mouvement brownien fractionnaire

L'étude de phénomènes irréguliers a pris une place très importante dans beaucoup de domaines scientifiques, comme la mécanique des fluides, le traitement de l'image et les mathématiques financières. L'auto-similarité est une propriété d'invariance qui génère de l'irrégularité. Par ailleurs, l'utilisation de fonctions aléatoires est un outil pratique pour obtenir des modèles irréguliers. A l'intersection de ces deux techniques se trouve le mouvement brownien fractionnaire.

Kolmogorov [60] est le premier à introduire le mouvement brownien fractionnaire (sous le nom "Wiener Spirals") en le définissant comme l'unique processus gaussien centré $B^H = (B_t^H, t \geq 0)$, de covariance

$$R^H(s,t) = E(B_s^H B_t^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right) \quad s,t \in \mathbb{R}_+.$$

où $H \in (0,1)$.

Plus tard, quand les articles de Hurst [48] et Hurst, Black et Simaika [49], consacrés à la capacité de stockage à long terme dans des réservoirs, ont été publiés, le paramètre H a pris le nom de "paramètre de Hurst".

Le calcul stochastique du mouvement brownien fractionnaire a débuté avec le travail novateur de Mandelbrot et Van Ness [71]. Ils ont considéré la représentation en moyenne mobile de B^H, via le processus de Wiener $(W_t; t \geq 0)$ sur un intervalle infini

$$B_t^H = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^{t} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW_s, \quad t \geq 0,$$

et ont appelé ce processus mouvement brownien frationnaire.

Remarquons que pour $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ est le mouvement brownien usuel. On a $E|B_s^H - B_t^H|^2 = |s-t|^{2H}$; ainsi B^H admet une version continue dont les trajectoires ne sont presque sûrement höldériennes que pour des indices strictement inférieurs à H. Par conséquent, plus H est petit plus les trajectoires sont irrégulières. Ce phénomène est dû au fait que les accroissements, qui sont stationnaires pour toutes les valeurs de H, sont positivement corrélés dans le cas $H > \frac{1}{2}$ et négativement corrélés dans le cas $H < \frac{1}{2}$, précisément :

$$Cov\left(B_h^H, B_{t+h}^H - B_t^H\right) \underset{t\to\infty}{\sim} H(2H-1)h^2 t^{2(H-1)}, \qquad \text{pour } H \neq \tfrac{1}{2} \text{ et } h > 0 \text{ fixés.} \tag{5}$$

Une propriété simple du mouvement brownien fractionnaire de paramètre H est son autosimilarité : le process $(B_{at}^H, t \geq 0)$ a la même loi que le processus $(a^H B_t^H, t \geq 0)$. Cette dernière propriété montre l'intérêt de ce processus pour les modélisations de fluctuations boursières, au trafic dans les réseaux de télécommunications (voir, par exemple [26]). De plus, plusieurs applications ont été trouvées en économie et sciences naturelles (voir, par exemple, Mandelbrot [70]).

Le processus B^H admet aussi une représentation intégrale de type (1) sur l'intervalle compact $[0,1]$(voir [27]) :

$$B_t = \int_0^t K_H(t,s) dW_s, \quad t \in [0,1] \tag{6}$$

où

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2}-H)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-(\frac{s}{u})^{\frac{1}{2}-H}\right)du \quad \text{si } H \leq \frac{1}{2}$$
$$K_H(t,s) = c_H(H-\frac{1}{2})\int_s^t (u-s)^{H-\frac{3}{2}}\left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{si } H > \frac{1}{2},$$

si $s < t$, et $K_H(t,s) = 0$ si $s \geq t$. Ici, c_H est la constante de normalisation

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}.$$

La propriété (5) implique que pour $H > \frac{1}{2}$ la somme des correlations diverge, i.e.

$$\sum_{n=1}^{\infty}\left|Cov\left(B_h^H, B_{nh}^H - B_{(n-1)h}^H\right)\right| = \infty \quad \text{pour tout } h > 0 \text{ fixé.}$$

Cette dernière propriété est connue sous le nom de dépendance à long terme (ou longue mémoire). Elle est souvent considérée comme une motivation pour étudier les processus fractionnaires. Dans le cas $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$, le processus gaussien B^H est ni un processus de Markov (voir [101] exercice (1.13) chapitre III), ni une semimartingale relativement à sa filtration naturelle.

Mouvement brownien bifractionnaire

Le mouvement brownien bifractionnaire est une généralisation du mouvement brownien fractionnaire. Rappelons que le mouvement brownien fractionnaire est le seul processus gaussien centré, à la fois autosimilaire et à accroissements stationnaires. Pour des accroissements assez petits, dans les applications telle que la turbulence, le mouvement brownien fractionnaire semble un excellent modèle, mais apparaît aussi inadéquat pour des accroissements larges. Pour cette raison, Houdré et Villa [47] ont introduit le mouvement brownien bifractionnaire comme une extension du mouvement brownien fractionnaire gardant certaines de ces propriétés (autosimilarité, stationnarité des accroissements assez petits, caractère gaussien) mais élargissant le kit d'outil de modélisation.

Le mouvement brownien bifractionnaire (mBbif) est un processus gaussien centré $B^{H,K} = (B_t^{H,K}, t \geq 0)$ de covariance

$$R^{H,K}(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right),$$

pour des indices $H \in (0,1)$ et $K \in (0,1]$. Dans le cas $K = 1$, $B^{H,1}$ est un mouvement brownien fractionnaire de paramètre de Hurst $H \in (0,1)$, noté B^H. En particulier, si $K = 1$ et $H = \frac{1}{2}$, $B^{\frac{1}{2},1}$ coïncide avec le mouvement brownien usuel.

Nous présentons maintenant un bref rappel des propriétés de base du mouvement brownien bifractionnaire.

Théorème 1. *Le mouvement brownien bifractionnaire $B^{H,K}$ possède les propriétés suivantes:*

1. *Auto-similarité:*

$$\left(B_{ct}^{H,K}, t \geq 0\right) \stackrel{d}{=} \left(c^{HK} B_t^{H,K}, t \geq 0\right) \text{ pout tout } c > 0.$$

2. *Propriété de quasi-hélice au sens de Kahane ([58], [59]): pour tout* $0 \leq s \leq t$,

$$2^{-K}|t-s|^{2HK} \leq E\left(\left|B_t^{H,K} - B_s^{H,K}\right|^2\right) \leq 2^{1-K}|t-s|^{2HK}. \tag{7}$$

3. *Continuité hölderienne :* $B^{H,K}$ *admet une version avec des trajectoires hölderiennes d'ordre* δ *pour tout* $\delta < HK$*; de plus elles ne sont pas dérivables.*

4. $B^{H,K}$ *n'est jamais un processus de Markov process ou une semimartingale, sauf quand c'est un mouvement brownien.*

5. *Si* $HK < \frac{1}{2}$ *(ou* $HK = \frac{1}{2}$ *avec* $K \neq 1$*),* $B^{H,K}$ *est un processus à courte mémoire, i.e.*

$$\sum_{j=m}^{\infty} E\left((B_{j+1}^{H,K} - B_j^{H,K})(B_{m+1}^{H,K} - B_m^{H,K})\right) < \infty, \text{ pour tout } m \geq 0.$$

6. *Si* $HK > \frac{1}{2}$*, la somme des corrélations diverge, i.e.*

$$\sum_{j=m}^{\infty} E\left((B_{j+1}^{H,K} - B_j^{H,K})(B_{m+1}^{H,K} - B_m^{H,K})\right) = \infty, \text{ pour tout } m \geq 0.$$

Cette dernière propriété de dépendance à long terme justifie l'étude du mouvement brownien bifractionnaire.

Ramarques : i) Il est clair que, pour $H \neq \frac{1}{2}$ et $K \in (0,1]$, $B^{H,K}$ n'est pas à accroissement stationnaire. En revanche, cette propriété est remplacée par celle de quasi-hélice (7).
ii) La propriété suivante de $B^{H,K}$ est intéressante : sa variation quadratique, dans le cas $2HK = 1$, est similaire à celle du mouvement brownien standard, i.e., $[B^{H,K}]_t = cst. \times t$; par conséquent ce cas est intéressant du point de vue de calcul stochastique.

Récemment, Lei et Nualart [64] ont montré que le mouvement brownien bifractionnaire $B^{H,K}$, admet se décompose en la somme d'un mouvement brownien fractionnaire de paramètre de Hurst HK plus un processus à trajectoires absolument continues :

$$\left(\sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} X_{t^{2H}}^K + B_t^{H,K}, t \geq 0\right) \stackrel{d}{=} \left(2^{\frac{1-K}{2}} B_t^{HK}, t \geq 0\right) \tag{8}$$

où

$$X_t = \int_0^{\infty} (1 - e^{-\theta t}) \theta^{-\frac{1+K}{2}} dW_\theta$$

avec W est un mouvement brownien indépendant de $B^{H,K}$.
Ce lien entre le mouvement brownien bifractionnaire et le mouvement brownien fractionnaire

conduit, en utilisant les résultats sur le mouvement brownien fractionnaire, à une meilleure compréhension et à des preuves simplifiées de quelques propriétés du mouvement brownien bifractionnaire qui ont été obtenues dans la littérature.

Chapitre 1 : Formules d'Itô et de Tanaka pour le mBbif multidimensionnel

Il est bien connu que les formules d'Itô et de Tanaka sont un outil puissant de l'analyse stochastique à cause de leur vaste domaine d'application. Ainsi, récemment, plusieurs chercheurs ont étudié des extensions des formules classiques d'Itô et de Tanaka aux processus de type (1) (voir, par exemple, [24] and [1]) ainsi qu'aux processus multidimensionnels comme le mouvement brownien multidimensionnel [114] et le mouvement brownien fractionnaire multidimensionnel [115].

Le chapitre 1 de cette thèse est constitué de la publication [39] en collaboration avec C. Tudor. Cette publication propose une extension des formules d'Itô et de Tanaka au mouvement brownien bifrationnaire unidimensionnel et multidimentionnel.

En supposant que $2HK \geq 1$, nous discutons les formules d'Itô et de Tanaka pour le mouvement brownien bifractionnaire. Dans le cas unidimensionnel, la formule d'Itô a déjà été établie dans [61] en utilisant une relation entre l'intégrale de Skorohord et une intégrale obtenue par une méthode de régularisation. Dans notre travail, nous avons proposé une approche différente, basée sur un développement de Taylor. Nous l'avons aussi utilisée dans le cas multidimensionnel. Précisément, nous avons obtenu le théorème suivant.

Théorème 2 (Itô unidimensionnel)**.** *Soit f une fonction de classe* C^2 *sur* $\mathbb{R}$ *telle que*

$$\max\{|f(x)|, |f'(x)|, |f''(x)|\} \leq ce^{\beta x^2}, \quad x \in \mathbb{R},$$

où c et β sont des constantes positives telles que $\beta < \frac{1}{4T^{2HK}}$. Supposons que $2HK \geq 1$. Alors $f'(B^{H,K}) \in Dom(\delta^{B^{H,K}})$ et, pour tout $t \in [0,T]$,

$$f\left(B_t^{H,K}\right) = f(0) + \int_0^t f'\left(B_s^{H,K}\right) \delta B_s^{H,K} + HK \int_0^t f''\left(B_s^{H,K}\right) s^{2HK-1} ds. \tag{9}$$

Traitons le cas de la formule de Tanaka. Comme dans le cas du mouvement brownien fractionnaire, le temps local L_t^x (à poids) de $B^{H,K}$ est défini de comme suit:

$$L_t^x = \lim_{\varepsilon \to 0} 2HK \int_0^t p_\varepsilon(B_s^{H,K} - x) s^{2HK-1} ds \ \text{ dans } L^2(\Omega), \tag{10}$$

où $p_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{y^2}{2\varepsilon}}$ est le noyau gaussien de variance $\varepsilon > 0$. Notons que L_t^x admet la représentation chaotique suivante :

$$L_t^x = 2HK \sum_{n=0}^{\infty} \int_0^t \frac{p_{s^{2HK}}(x)}{s^{(n-2)HK+1}} H_n\left(\frac{x}{s^{HK}}\right) I_n(1_{[0,s]}^{\otimes n}) ds \tag{11}$$

où I_n représente l'intégrale multiple par rapport au mBbif et H_n est le nième polynôme d'Hermite. La combinaison de la décomposition (11) et du théorème 2 est à la base de la preuve du théorème suivant.

Théorème 3 (Tanaka unidimensionnel)**.** *Soit* $\left(B_t^{H,K}, t \in [0,T]\right)$ *un mouvement brownien bifractionnaire avec* $2HK \geq 1$. *Alors, pour tout* $x \in \mathbb{R}$, *on a* $sign(B_\cdot^{H,K} - x) \in Dom(\delta^{B^{H,K}})$ *et, pour tout* $t \in [0,T]$ *et* $x \in \mathbb{R}$, *on a*

$$\left|B_t^{H,K} - x\right| = |x| + \int_0^t sign(B_s - x)\delta B_s^{H,K} + L_t^x, \tag{12}$$

où

$$sign(x) = \begin{cases} 1 & \text{si } x > 0 \\ -1 & \text{si } x \leq 0. \end{cases}$$

Soient deux vecteurs $H = (H_1, \ldots, H_d) \in [0,1]^d$ et $K = (K_1, \ldots, K_d) \in (0,1]^d$. Nous introduisons le mouvement brownien bifractionnaire d-dimensionnel

$$B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$$

comme un vecteur gaussien centré dont les composantes sont des mouvements browniens bifractionnaires unidimensionnels indépendants.
Nous étendons la formule d'Itô au cas multidimensionnel.

Théorème 4 (Itô multidimensionnel)**.** *Soit* $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ *un mouvement brownien bifractionnaire d-dimensionnel, et soit* f *une fonction de classe* $\mathrm{C}^2\left(\mathbb{R}^d, \mathbb{R}\right)$ *telle que, pour tout* $x \in \mathbb{R}^d$,

$$\max_{1 \leq i,l \leq d}\left(|f(x)|, |\frac{\partial f}{\partial x_i}(x)|, |\frac{\partial^2 f}{\partial x_i \partial x_l}(x)|\right) \leq ce^{\beta|x|^2},$$

où c *et* β *sont des constantes positives telles que* $\beta < \frac{1}{4T^{2(HK)^*}}$ *où* $(HK)^* = \max_{1 \leq i \leq d} H_iK_i$. *Nous supposons que* $2H_iK_i > 1$ *pour certain* $i = 1, ..., n$. *Alors pour tout* i *nous avons* $\frac{\partial f}{\partial x_i}\left(B^{H_i,K_i}\right) \in Dom(\delta^{B_s^{H_i,K_i}})$ *et pour tout* $t \in [0,T]$,

$$f\left(B_t^{H,K}\right) = f(0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}\left(B_s^{H_i,K_i}\right)\delta B_s^{H_i,K_i} + \sum_{i=1}^d H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(B_s^{H,K}\right)s^{2H_iK_i-1}ds. \tag{13}$$

Dans le théorème de Tanaka unidimensionnel, on utilise la fonction $|z|$ qui est la fonction noyau du potentiel Newtonien unidimensionnel, i.e. $\frac{1}{2}\Delta|z| = \delta(z)$ et que $\nabla|z| = sign(z)$. Intuitivement, dans le cas $d-$dimensionnel, on vas remplacer $|z|$ et $sign(z)$ dans (12) respectivement par $U(z)$ et $\nabla U(z)$; où $U(z)$ est la fonction noyau du potentiel Newtonien $d-$dimensionnel si $d \geq 3$ et le potentiel logarithmique si $d = 2$, i.e.,

$$U(z) = \begin{cases} -\frac{\Gamma(d/2-1)}{2\pi^{d/2}}\frac{1}{|z|^{d-2}} & \text{si } d \geq 3 \\ \frac{1}{\pi}log|z| & \text{si } d = 2. \end{cases}$$

Définissons

$$\bar{U}(s,z) = \frac{1}{\prod_{j=1}^d \sqrt{2H_jK_j}}s^\theta U\left(\frac{(z_1 - x_1)}{\sqrt{2H_1K_1}}s^{1/2-H_1K_1}, ..., \frac{(z_d - x_d)}{\sqrt{2H_dK_d}}s^{1/2-H_dK_d}\right) \tag{14}$$

où $x = (x_1, ..., x_d) \in \mathbb{R}^d$ et $0 < \gamma := \frac{1}{2}(2-d) + \theta + (d-2)(HK)^* - \sum_{i=1}^d H_i K_i$, avec $(HK)^* = \max\{H_1K_1, \ldots, H_dK_d\}$.

Nous obtenons la formule de Tanaka suivante :

Théorème 5. *Soit $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ un mBbif d-dimensionnel tel que $2H_iK_i > 1$ pour tout $i = 1, ...d$. Alors la formule suivante est satisfaite sur l'espace de Sobolev $D^{\alpha-1,2}$ pour un certain $\alpha < \frac{1}{2(HK)^*} - d/2$:*

$$\bar{U}(t, B_t^{H,K}) = \bar{U}(0,0) + \int_0^t \partial_s \bar{U}(s, B_s^{H,K}) ds + \sum_{i=1}^d \int_0^t \frac{\partial \bar{U}(s, B_s^{H,K})}{\partial x_i} \delta B_s^{H_i,K_i} + L^\theta(t,x) \qquad (15)$$

où le temps local (à poids) généralisé $L^\theta(t,x)$ est défini par

$$L^\theta(t,x) = \sum_{n=(n_1,\ldots,n_d)} \int_0^t \prod_{i=1}^d \frac{p_{s^{2H_iK_i}}(x_i)}{s^{\frac{1}{2}+(n_i-1)H_iK_i}} H_{n_i}\left(\frac{x_i}{\sqrt{s^{2H_iK_i}}}\right) I_{n_i}^i(1_{[0,s]}^{\otimes n_i}) s^\theta ds.$$

Chapitre 2 : Densités d'occupation pour certains processus en relation avec le mouvement brownien fractionnaire

Le chapitre 2 a pour objectif, en se basant sur les techniques de calcul de Malliavin, d'établir l'existence d'une densité d'occupation de carré intégrable pour deux classes de processus stochastiques. Premièrement nous considérons un processus gaussien avec dérive aléatoire absolument continue et, deuxièmement, nous traîtons le cas d'une intégrale de Skorohod par rapport au mouvement brownien fractionnaire de paramètre de Hurst $H > \frac{1}{2}$. Ces résultats font l'objet d'une publication soumise [36] en collaboration avec D. Nualart, Y. Ouknine et C. Tudor.

Soit $x : [0,1] \to \mathbb{R}$ une fonction mesurable. La mesure d'occupation de x est définie de la façon suivante :

$$\mu(x)(C) = \int_0^1 \mathbf{1}_C(x_s) ds,$$

où C est un sous-ensemble borélien de $\mathbb{R}$. On dit que x admet une densité d'occupation par rapport à la mesure de Lebesgue λ si la mesure μ est absolument continue par rapport à λ; dans ce ca, la densité d'occupation de la fonction x est définie comme la dérivée de Radon-Nikodym $\frac{d\mu}{d\lambda}$. Pour un processus continu $X = \{X_t, t \in [0,1]\}$, on dit que X a une densité d'occupation (ou un temps local) sur $[0,1]$ si, pour presque tout $\omega \in \Omega$, $X(\omega)$ a une densité d'occupation sur $[0,1]$.

Les temps locaux des semimartingales ont été largement étudiés (voir par exemple, la monographie [101]). De même, les temps locaux des processus gaussiens ont aussi fait l'objet d'une riche littérature (voir Marcus et Rosen [74]).

Berman [14] a montré l'existence d'un effet de régularité inverse entre le temps local et le processus associé. Cette observation a fait du temps local un outil puissant pour étudier les trajectoires irrégulières d'un processus continu.

En général, les trajectoires d'un processus anticipant, en particulier celles du processus associé à une intégrale de Skorohod, sont très irrégulières. Ceci conduit à étudier leur temps local (ou densité d'occupation). Dans cet esprit, différentes méthodes ont été utilisées pour étudier les densités d'occupation des processus anticipants. En se basant sur l'idée de Berman [13] qui utilise l'analyse de Fourier, Imkeller ([51], [52] et [53]) a donné un critère pour l'existence d'un temps local de carré intégrable pour un processus intégral de Skorohod qui vit dans le second chaos de Wiener. Un autre critère général, pour l'existence d'un temps local pour la vaste classe des processus anticipants, qui ne sont en général ni des semimartingales, ni des processus gaussiens a été établi par Imkeller et Nualart [50]. La preuve de ce dernier résultat combine les techniques du calcul de Malliavin avec le critère donné par Geman et Horowitz [44], qui ont étudié le cas du mouvement brownien avec dérive anticipante, et le cas d'un processus associé à une intégrale de Skorohod.

Le but de ce chapitre est d'établir l'existence de la densité d'occupation pour deux classes de processus stochastiques qui ont des relations avec le mouvement brownien fractionnaire. On utilise l'approche introduite par Imkeller et Nualart [50]. Nous commençons par le cas d'un processus $X = \{X_t, t \in [0,1]\}$ de la forme

$$X_t = B_t + \int_0^t u_s ds,$$

où B est gaussien et où u est un processus stochastique mesurable par rapport à la tribu engendrée par B. Nous supposons que la variance de l'accroissement du processus gaussien B sur un intervalle $[s,t]$ se comporte comme $|t-s|^{2\rho}$, pour un certain $\rho \in (0,1)$. Ceci inclut, par exemple, le mouvement brownien bifractionnaire de paramètres $H \in (0,1)$ et $K \in (0,1]$ et donc aussi le mouvement brownien fractionnaire (cas particulier où $K = 1$). Sous des hypothèses raisonnables de régularité pour le processus u, nous montrons l'existence d'une densité d'occupation de carré integrable par rapport à la mesure de Lebesgue pour le processus X.

Notre deuxième exemple est un processus sous forme divergence $X = \{X_t, t \in [0,1]\}$, par rapport au mouvement brownien fractionnaire B de paramètre de Hurst $H \in (\frac{1}{2}, 1)$, de la forme :

$$X_t = \int_0^t u_s \delta B_s^H.$$

Nous fournissons des conditions d'intégrabilité sur u et ses dérivées itérées au sens de Malliavin qui assurent l'existence d'une densité d'occupation de carré intégrable pour X.

Processus de Rosenblatt

L'objet de cette section est de donner une brève introduction aux processus d'Hermite, en particulier au processus de Rosenblatt.

Considérons une suite stationnaire centrée réduite gaussienne $(X_n)_{n\geq 1}$ c'est-à-dire que $EX_1 = 0$ et $EX_1^2 = 1$.
Soit G une fonction réelle qui vérifie $EG(X_1) = 0$ et $E(G(X_1))^2 < \infty$. Alors G admet un développement d'Hermite dans l'espace $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx)$, qui est de la forme

$$G(x) = \sum_{j=1}^{+\infty} a_j H_j(x), \text{ où } a_j = E\left(G(X_1)H_j(X_1)\right).$$

Notons par $r(n)$ la fonction de covariance de $(X_n)_{n\geq 1}$ et par k le rang de Hermite de G, i.e. $k = \min\{j; a_j \neq 0\}$. Supposons que

$$r(n) := E(X_1X_n) = n^{\frac{2H-2}{k}}L(n), \text{ et } H \in (\frac{1}{2}, 1)$$

où L est une fonction à variation lente à l'infini, i.e. L est bornée sur les intervalles finis et pour tout $t > 0$

$$\frac{L(tx)}{L(x)} \longrightarrow 1, \quad \text{quand } x \longrightarrow +\infty.$$

Le processus d'Hermite est apparu pour la première fois dans le théorème non-central limite suivant, prouvé par Taqqu [108] (voir aussi Dobrushin et Major [32]). Le processus

$$\frac{1}{n^H}\sum_{j=1}^{[nt]} G(X_j)$$

converge (au sens des lois fini-dimensionnelles) lorsque $n \longrightarrow \infty$ vers le processus

$$Z_t^{k,H} = C(H,k)\int_{\mathbb{R}^k}\int_0^t \left(\Pi_{i=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right) ds dW_{y_1}\dots dW_{y_k}$$

où $x_+ = \max(x, 0)$ et $(W_t; t \in [0,T])$ est un mouvement brownien.

Définition. Le processus $Z^{k,H} = \left(Z_t^{k,H}; t \in [0,T]\right)$ est appelé processus d'Hermite d'ordre k et de paramètre H. Pour $k = 1$, on retrouve le mouvement brownien fractionnaire, et pour $k = 2$, il s'agit du processus de Rosenblatt.

Le processus d'Hermite $Z^{k,H}$ vérifie les propriétés suivantes:

1. $Z^{k,H}$ est un processus centré tel que $Var(Z_1^{k,H}) = 1$.

2. Ses accroissements sont stationnaires.

3. La fonction de covariance de $Z^{k,H}$ est donnée par

$$Cov(Z_t^{k,H}, Z_s^{k,H}) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad t, s \in [0,T];$$

   par conséquent

$$E\left|Z_t^{k,H} - Z_s^{k,H}\right|^2 = |t-s|^{2H}.$$

4. Si $k \geq 2$, alors $Z^{k,H}$ est non-gaussien; le processus de Hermite $Z^{k,H}$ d'ordre k vit dans le chaos de Wiener d'ordre k de W.

5. $Z^{k,H}$ admet une version continue dont les trajectoires sont presque sûrement höldériennes pour tous les indices strictement inférieurs à H. Grâce au théorème de continuité de Kolmogorov, c'est une conséquence de la propriété 3 et de l'équivalence des normes L^p dans un chaos d'orde fixé.

6. $Z^{k,H}$ a une longue mémoire (dépendance à long terme). Plus précisément, sa fonction de covariance $Cov(Z_1^{k,H}, Z_{n+1}^{k,H} - Z_n^{k,H})$ se comporte comme n^{2H-2} à l'infini.

Exceptée la propriété 6 concernant le caractère gaussien, nous remarquons que les propriétés 1 à 5 du processus d'Hermite $Z^{k,H}$ sont similaires à celles du mouvement brownien fractionnaire de paramètre de Hurst $H > \frac{1}{2}$.
Tudor a établi dans [111] qu'un processus de Hermite $Z^{k,H}$ d'ordre $k \geq 1$ peut être représenté, en loi, comme une intégrale multiple itérée par rapport au processus de Wiener usuel. Ce résultat fait l'objet de la proposition suivante.

Proposition 1. *Fixons $k \geq 1$ et $H > \frac{1}{2}$. Le processus de Hermite $Z^{k,H}$ d'ordre k et de paramètre H a la même loi que le processus*

$$d(H) \int_0^t \ldots \int_0^t dW_{y_1} \ldots dW_{y_k} \int_{y_1 \vee y_2 \ldots \vee y_k}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \ldots \frac{\partial K^{H'}}{\partial u}(u, y_k) du; \ t \in [0, T] \tag{16}$$

où $(W_t, t \in [0, T])$ est un mouvement brownien,

$$H' := 1 + \frac{H-1}{k}; \qquad d(H) := \frac{1}{H+1} \left(\frac{2(2H-1)}{H} \right)^{\frac{1}{2}},$$

et $K^{H'}$ le noyau standard défini dans (6).

Récemment, la représentation (16) a été utilisée par plusieurs auteurs pour développer l'analyse stochastique pour les processus d'Hermite (voir [111], [79], [75], [112], [18], [22]).

Chapitre 3 : Théorème non-central limite pour la variation cubique d'une classe des processus stochastiques autosimilaire

Nous nous intéressons, dans ce chapitre, au comportement asymptotique, quand $N \longrightarrow \infty$, de la variation cubique du processus de Rosenblatt $Z^{(H)}$ définie par

$$V^{3,N}(Z^{(H)}) = \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}} \right)^3}{E \left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}} \right)^3} - 1 \right).$$

Pour les processus autosimilaires (en particulier le processus de Rosenblatt), l'étude de leurs variations constitue un outil fondamental pour construire des estimateurs du paramètre

d'autosimilarité. Les processus autosimilaires sont des modèles convenables à la modélisation de nombreux phénomènes, où la longue mémoire est un facteur important. Cela inclu le trafic internet (cf. [117]), l'hydrologie (cf. [76]) et l'économie (cf. [69], [118]). La tâche de modélisation la plus importante est ensuite d'estimer le paramètre d'autosimilarité, parce qu'il caractérise toute l'importance des propriétés de dépendance à long terme du processus.
Il existe une connection directe entre le comportement des variations et la convergence d'un estimateur statistique pour l'indice d'autosimilarité (voir [23], [112]).
Le cas de la variation quadratique du processus de Rosenblatt $(Z_t^{(H)})_{t\in[0,1]}$ d'indice $H > \frac{1}{2}$, avec un horizon de temps fini $[0,1]$, a été étudié par Tudor et Viens dans [112].
Dans le cas d'un mouvement brownien fractionnaire B^H, la non-normalité de la variation quadratique lorsque $H \in (\frac{3}{4}, 1)$ peut être évitée en utilisant soit les "longer filters" (c'est à dire on remplace les accroissements $B^H_{\frac{i+1}{N}} - B^H_{\frac{i}{N}}$ par $B^H_{\frac{i+1}{N}} - 2B^H_{\frac{i}{N}} + B^H_{\frac{i-1}{N}}$), soit des variations d'ordre grand. Dans notre travail, nous avons considéré le deuxième choix : nous remplaçons la variation quadratique par la variation cubique. Dans le cas de B^H, ceci n'a pas de sens puisque le troisième moment d'une variable aléatoire gaussienne est nul. Pour étudier la variation cubique du processus $Z^{(H)}$, nous avons utiliser la décomposition chaotique de Wiener pour la statistique $V^{3,N}(Z^{(H)})$ et nous l'avons décomposée en plusieurs termes qui appartiennent aux chaos d'ordre 2, 4 et 6. En normalisant par N^{1-H}, nous avons montré que $E\left(N^{1-H}V^{3,N}(Z^{(H)})\right)^2$ converge vers une constante $C(H)$ quand $N \longrightarrow \infty$. Ensuite, pour étudier la loi limite nous avons utilisé le critère suivant:

Théorème 6 (Nualart-Ortiz−Latorre)**.** *Fixons $n \geq 2$. Soit $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ une suite de variables aléatoires dans le nème chaos de Wiener telle que $EF_k^2 \longrightarrow 1$ lorsque $k \longrightarrow \infty$. Alors*

$$(F_k)_{k\geq 0} \text{ converge en loi vers une loi normal } \mathcal{N}(0,1).$$
$$\Longleftrightarrow$$
$$\|DF_k\|^2_{\mathcal{H}} \longrightarrow n \text{ dans } L^2(\Omega) \text{ quand } k \longrightarrow \infty.$$

Comme dans [112], [22], le terme dominant noté T_N de la décomposition de $V^{3,N}(Z^{(H)})$ est celui qui vit dans le deuxième chaos et qui doit être normalisé par N^{1-H} pour avoir une limite non triviale. Nous avons établi que $\|N^{1-H}DT_N\|^2_{\mathcal{H}} \longrightarrow c > 2$ dans $L^2(\Omega)$ quand $N \longrightarrow \infty$. Ce qui implique que la loi limite de $N^{1-H}V^{3,N}(Z^{(H)})$ est non-normale. De plus et comme dans le cas de la variation quadratique nous avons obtenu la même limite qui est, à une constante près, une variable aléatoire de Rosenblatt d'indice H. Ce resultat fait l'objet d'une publication [41] soumise en collaboration avec C. Tudor.

Chapitre 4 : Estimation de la dérive de mouvement brownien fractionnaire

Soit $B^H = \left\{\left(B_t^{H,1}, ..., B_t^{H,d}\right); t \in [0,T]\right\}$ un mouvement brownien fractionnaire (mbf) d-dimensionnel de paramètre $H \in (0,1)$, défini sur un espace de probabilité $(\Omega, \mathcal{F}, P)$. Pour chaque $i = 1, \ldots, n$,

$(\mathcal{F}_t^i)_{t\in[0,T]}$ dénote la filtration engendrée par $\left(B^{H,i}\right)_{t\in[0,T]}$.
Soit M un sous espace de l'espace de Cameron-Martin défini par

$$M = \Big\{ \varphi : [0,T] \to \mathbb{R}^d ; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ avec } \dot{\varphi}^i \in L^2([0,T]) \\ \text{et } \varphi^i \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T])\right), i = 1, ..., d \Big\}$$

où $I_{0+}^{H+\frac{1}{2}}$ est l'intégral fractionnaire à gauche de Riemann-Liouville d'ordre $(H+\frac{1}{2})$.

Soit $\theta = \left\{(\theta_t^1, \ldots, \theta_t^d); t\in[0,T]\right\}$ une fonction de M. Alors, en appliquant le théorème de Girsanov (voir [84]), il existe une mesure de probabilité P_θ absolument continue par rapport à P sous laquelle le processus $\widetilde{B}^H$ défini par

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t\in[0,T]$$

est un mbf centré de paramètre H. Autrement dit, sous la probabilité P_θ, le processus B^H est un mbf de dérive θ.

Nous nous considérons dans ce chapitre le problème de l'estimation de la dérive θ de B^H sous la probabilité P_θ, dans le cas où $H < 1/2$. Nous étudions l'estimation de θ sous le risque quadratique usuel, qui est défini pour tout estimateur $\delta = \{(\delta_t^1, \ldots, \delta_t^d),\ t\in[0,T]\}$ de θ par

$$\mathcal{R}(\theta,\delta) = E_\theta\left[\int_0^T ||\delta_t - \theta_t||^2 dt\right]$$

où E_θ est l'espérance relativement à la probabilité P_θ.
Un estimateur δ de θ est dit sans biais si, pour tout $t\in[0,T]$

$$E_\theta(\delta_t^i) = \theta_t^i, \qquad i = 1,\ldots,d$$

et il est adapté si, pour chaque $i = 1,\ldots,d$, δ^i est adapté à $\left(\mathcal{F}_t^i\right)_{t\in[0,T]}$.

Récemment, Privault et Reveillac dans [96] ont construit, dans un cadre infini dimensionnel, des estimateurs sans biais de la dérive $(\theta_t)_{t\in[0,T]}$ pour une martingale gaussienne $(X_t)_{t\in[0,T]}$ de variation quadratique $\sigma_t^2 dt$, où $\sigma \in L^2([0,T], dt)$ est fonction non nulle. Précisément, ils ont montré que $\hat{\theta} = (X_t)_{t\in[0,T]}$ est un estimateur efficace de $(\theta_t)_{t\in[0,T]}$. D'autre part, à l'aide de calcul de Malliavin, ils ont construit des estimateurs suroptimaux de la dérive d'un processus gaussien de la forme:

$$X_t := \int_0^t K(t,s) dW_s, \qquad t\in[0,T],$$

où $(W_t)_{t\in[0,T]}$ est un mouvement brownien et $K(.,.)$ est noyau déterministe. Ces estimateurs sont biaisés de la forme $X_t + D_t \log F$, où F est un surharmonique variable aléatoire et D la dérivée au sens de Malliavin.

Dans ce chapitre, nous utilisons des techniques basées sur le théorème tout d'abord de Girsanov du mbf et le calcul fractionnaire pour établir que $\widehat{\theta} = B^H$est un estimateur efficace de θ

sous la probabilité P_θ de risque

$$\mathcal{R}(\theta, B^H) = E_\theta\left[\int_0^T \|B_t^H - \theta_t\|^2 dt\right] = \frac{T^{2H+1}}{2H+1}d.$$

De plus, nous nous montrons que $\hat{\theta} = B^H$ est un estimateur de maximum de vraisemblance de θ.

D'autre part, nous nous construisons une classe des estimateurs biaisés suroptimaux de type James-Stein de la forme:

$$\delta(B^H)_t = \left(1 - at^{2H}\left(\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right)\right) B_t^H, \qquad t \in [0, T].$$

Nous nous donnons des conditions suffisantes sur la fonction r et sur la constante a pour que $\delta(B^H)$ domine B^H sous le risque quadratique usuel i.e.

$$\mathcal{R}\left(\theta, \delta\left(B^H\right)\right) < \mathcal{R}\left(\theta, B^H\right) \qquad \text{for all } \theta \in M.$$

Ce chapitre fait l'objet d'une publication [37] en collaboration avec I. Ouassou et Y. Ouknine.

Calcul de Malliavin sur l'espace canonique de Lévy : l'approche de Solé et al. [106]

Sur l'espace de Poisson et d'une façon générale, deux approches du calcul de variations ont été introduites : l'approche variationnelle (voir par exemple Bichteler et al. [15] et Carlen et Pardoux [21]), et l'approche chaotique (voir par exemple Nualart et Vives [85] et León et Tudor [65]). Depuis ces dernières années, la théorie du calcul de Malliavin a été étendue dans un cadre plus général à l'espace de Lévy par plusieurs approches, avec pour motivations des applications en finance (voir par exemple Løkka [66] , Di Nunno et al. [31] et Solé et al. [106]).

Un processus de Lévy est un processus stochastique à accroissements indépendants et stationnaires. Si $(X_t, t \geq 0)$ est un processus de Lévy, alors $X_t - X_s$ avec $t \geq s$ est indépendant de l'histoire du processus avant le temps s et sa loi ne dépend pas de t ou s séparément, mais seulement de $t - s$.

Nous considérons un processus de Lévy $X = (X_t, 0 \leq t \leq 1)$ défini sur un espace de probabilité $\big(\Omega, (F_t^X)_{0\leq t\leq 1}, P\big)$, où $(F_t^X)_{0\leq t\leq 1}$ est la filtration engendrée par X. Alors, il existe un triplet (γ, σ^2, ν): où $\gamma \in \mathbb{R}$, $\sigma \geq 0$ et une mesure $\nu(dz)$ sur $\mathbb{R}$ appelée mesure de Lévy tels que $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} 1 \wedge x^2 \nu(dx) < \infty$ et

$$E\left(\exp(itX_1)\right) = \exp\left\{i\gamma t - \frac{1}{2}\sigma t^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx1_{\{|x|\leq 1\}}\right)\nu(dx)\right\}, \qquad \forall t \in \mathbb{R}.$$

Dans tout ce chapitre nous supposons que $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$.

Il est bien connu que le processus X admet une représentation de Lévy-Itô :

$$X_t = \gamma t + \sigma W_t + \int\int_{(0,t]\times\{|x|>1\}} xN(ds, dx) + \lim_{\varepsilon \downarrow 0} \int\int_{(0,t]\times\{\varepsilon<|x|\leq 1\}} x\widetilde{N}(ds, dx)$$

où W est un mouvement brownien standard, N est la mesure des sauts de X définie pour tout borélien $E \in B([0,1] \times \mathbb{R} - \{0\})$ par :

$$N(E) = \#\{t : (t, \Delta X_t) \in E\},$$

où $\Delta X_t = X_t - X_{t^-}$, $\#$ dénote le cardinal et $\widetilde{N}$ est la mesure des sauts compensée :

$$\widetilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx).$$

Suivant l'approche d'Itô [55], X peut être étendu à une mesure aléatoire sur $([0,1] \times \mathbb{R}, B([0,1] \times \mathbb{R}))$:

$$M(E) = \sigma \int_{\{s \in [0,1] : (s,0) \in E\}} dW_s + \lim_{n \to \infty} \int \int_{\{(s,x) \in E : \frac{1}{n} < |x| < n\}} x\widetilde{N}(ds, dx)$$

pour tout $E \in B([0,1] \times \ \mathbb{R})$ tel que $\mu(E) < \infty$, où μ est une mesure $\sigma-$finie sur $[0,1] \times \mathbb{R}$:

$$\mu(E) = \sigma^2 \int_{\{s \in [0,1] : (s,0) \in E\}} ds + \int \int_{\{E - \{s \in [0,1] : (s,0) \in E\} \times \{0\}\}} x^2 ds\nu(dx).$$

M est appelée mesure à valeur martingale de type $(2, \mu)$. Le second moment existe toujours et peut s'exprimer en fonction de la mesure μ (voir Applebaum [3]). De plus M est une mesure aléatoire indépendante centrée:

$$E\left(M(E_1)M(E_2)\right) = \mu(E_1 \cap E_2)$$

pour tout $E_1, E_2 \in B([0,1] \times \mathbb{R})$ tels que $\mu(E_1) < \infty$ et $\mu(E_2) < \infty$.
Utilisant la mesure aléatoire M, comme sur l'espace de Wiener [81], on peut construire l'intégrale multiple $I_n(f)$ par rapport au processus de Lévy comme une isométrie entre $L^2(\Omega)$ et l'espace $L_n^2 = L^2\left(([0,1] \times \mathbb{R})^n, B(([0,1] \times \mathbb{R})^n), \mu^{\otimes n}\right)$. En effet, on débute en considérant le cas élémentaire:

$$f = 1_{E_1 \times \ldots \times E_n}$$

où $E_1, \ldots, E_n \in B([0,1] \times \mathbb{R})$ sont disjoints deux à deux et $\mu(E_i) < \infty$ pour tout i. On définit alors $I_n(f) = M(E_1) \ldots M(E_n)$. Ensuite, on prolonge $I_n(f)$ à tout L_n^2 par linéarité et continuité. On obtient ainsi la propriété de représentation chaotique suivante :

$$L^2(\Omega, F^X, P) = \oplus_{n=0}^{\infty} I_n(L_n^2).$$

Par conséquence, toute variable aléatoire $F \in L^2(\Omega, F^X, P)$, peut être représentée sous la forme

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$$

où $f_n \in L_n^2$. A ce stade, et comme dans les cas brownien et poissonien, on peut introduire un calcul de Malliavin pour les processus de Lévy. Si

$$\sum_{n=0}^{\infty} n n! \|f_n\|_{L_n^2}^2 < \infty$$

alors la dérivée de Malliavin de F est introduite par

$$(z,w) \in ([0,1]\times\mathbb{R})\times\Omega \xrightarrow{DF} D_zF(w) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(z,.)).$$

Le domaine de l'opérateur de derivation D est défini par:

$$D^{1,2} = \left\{ F = \sum_{n=0}^{\infty} I_n(f_n) : \sum_{n=0}^{\infty} nn!\|f_n\|^2_{L^2_n} < \infty \right\}.$$

Notons par $D^{k,2}$, $k \geq 1$, le domaine de la $k$ième dérivée itérée $D^{(k)}$, qui est un espace de Hilbert muni du produit scalaire

$$\langle F,G\rangle = E(FG) + \sum_{j=1}^{k} E\int_{([0,1]\times\mathbb{R})^j} D_z^{(j)}FD_z^{(j)}G\mu(dz).$$

Maintenant, on peut définir l'opérateur adjoint de D, noté δ et appelé opérateur de divergence ou intégrale de Skorohod. Soit $u \in H = L^2\left([0,1]\times\mathbb{R}\times\Omega, B([0,1]\times\mathbb{R})\otimes F_T^X, \mu\otimes P\right)$. Alors, pour tout $z \in [0,1]\times\mathbb{R}$, $u(z)$ admet la représentation suivante

$$u(z) = \sum_{n=0}^{\infty} I_n(f_n(z,.)), \tag{17}$$

où on a $f_n \in L^2(([0,1]\times\mathbb{R})^{n+1}, \mu^{\otimes^{n+1}})$ et f_n est symétrique en ses n dernières variables. Si

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{n+1} < \infty$$

où $\tilde{f}_n$ est la symétrisation de f_n, dans ce cas, l'intégrale de Skorohod $\delta(u)$ de u est définie par

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

Le domaine de δ est l'ensemble des processus de type (17) satisfaisants

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{n+1} < \infty.$$

En outre, on obtient la formule d'intégration par parties

$$E(F\delta(u)) = E\int\int_{[0,1]\times\mathbb{R}} D_zFu(z)\mu(dz), \quad F \in D^{1,2}.$$

Nous utiliserons les notations suivantes

$$\delta(u) = \int_0^1\int_{\mathbb{R}} u_z\delta M(dz) = \int_0^1\int_{\mathbb{R}} u_{s,x}\delta M(ds,dx).$$

Dans le cas où le processus u est adapté, Solé et al. [106] ont montré que l'intégrale de Skorohod coïncide avec l'intégrale semimartingale dirigé par M introduit dans [3].

Pour $k \geq 1$, notons par $L^{k,2}$ l'espace $L^2(([0,1]\times\mathbb{R}; D^{k,2}), \mu)$. En particulier, on peut montrer que l'espace $L^{1,2}$ coïncide avec l'ensemble des u de type (17) tel que

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|_{n+1}^2 < \infty.$$

On a aussi $L^{k,2} \subset Dom(\delta)$ pour $k \geq 1$ et, pour tout $u, v \in L^{1,2}$,

$$E(\delta(u)\delta(v)) = E\int\int_{[0,1]\times\mathbb{R}} u(z)v(z)\mu(dz) + E\int\int_{([0,1]\times\mathbb{R})^2} D_z u(z')D_{z'}v(z)\mu(dz)\mu(dz').$$

En particulier

$$E(\delta(u))^2 = E\int\int_{[0,1]\times\mathbb{R}} u(z)^2\mu(dz) + E\int\int_{([0,1]\times\mathbb{R})^2} D_z u(z')D_{z'}u(z)\mu(dz)\mu(dz').$$

La relation de commutation entre l'opérateur de dérivation et l'opérateur de divergence est comme suit. Soit $u \in L^{1,2}$ tel que $D_z u \in Dom(\delta)$. Alors $\delta(u) \in D^{1,2}$ et

$$D_z\delta(u) = u(z) + \delta(D_z(u)), \quad z \in [0,1]\times\mathbb{R}.$$

Chapitre 5 : Processus intégral d'Itô-Skorohod sur l'espace canonique de Lévy

Le chapitre 5 de cette thèse porte sur l'étude des processus intégraux d'Itô-Skorohod (au sens de [111]) sur l'espace canonique de Lévy. L'étude du lien entre l'intégrale de Skorohod et l'intégrale d'Itô a été présentée par Tudor [111] sur l'espace de Wiener et par Peccati et Tudor [93] sur l'espace de Poisson. Cette étude admet des applications en finance (voir Tudor [110]). En utilisant la nouvelle approche du calcul de Malliavin pour les processus de Lévy introduite dans [106], nous avons pu généraliser ce type d'intégrale de Itô-Skorohod aux processus de Lévy.

L'objectif de ce chapitre est d'utiliser le calcul de Malliavin sur l'espace canonique de Lévy, développé par Solé et al. [106], pour étudier la relation entre des processus anticipés de type intégrale de Skorohod et des processus de type intégrale d' Itô-Skorohod (dans le sens de [111] et [93]).

Comme dans le cas brownien, nous avons établi les propriétés suivantes:

1. Soit $f \in L_s^2(([0,1]\times\mathbb{R})^n, \mu^{\otimes^n})$ et $A \in B([0,1])$. Alors

$$E\left(I_n(f)/F_A^X\right) = I_n(f1_{(A\times\mathbb{R})}^{\otimes^n})$$

   où $F_A^X = \sigma(X_t - X_s : s,t \in A)$.

2. Supposons que $F \in D^{1,2}$ et $A \in B([0,1])$. Alors l'espérance conditionnelle $E(F/F_A^X)$ appartient à $D^{1,2}$ et pour tout $z \in [0,1]\times\mathbb{R}$

$$D_z E\left(F/F_A^X\right) = E\left(D_z F/F_A^X\right)1_{A\times\mathbb{R}}(z).$$

Par conséquent, nous obtenons la formule de Clark-Ocone correspondante.

Proposition 2 (Formule de Clark-Ocone-Haussman généralisée). *Soit $F \in D^{1,2}$. Alors pour tout $0 \le s < t \le 1$, nous avons*

$$F = E\left(F/F^X_{(s,t]^c}\right) + \delta(h_{s,t}(\cdot))$$

où pour $(r,x) \in [0,1] \times \mathbb{R}$ nous notons $h_{s,t}(r,x) = E\left(D_{r,x}F/F_{(r,t]^c}\right) 1_{(s,t]^c}(r)$. De plus

$$F = E\left(F/F^X_{(s,t]^c}\right) + \int\int_{(s,t]\times R} {}^{(p,t)}(D_zF)\ dM_z$$

$$= E\left(F/F^X_{(s,t]^c}\right) + \sigma \int_s^t {}^{(p,t)}(D_{r,0}F)dW_r + \int\int_{(s,t]\times R_0} {}^{(p,t)}(D_{r,x}F)\ \widetilde{N}(dr,dx)$$

où ${}^{(p,t)}(DF)$ est la projection prévisible de DF par rapport à la filtration $\left(F^X_{(r,t]^c}\right)_{r\le t}$.

A partir de ces résultats ci-dessus, nous avons prouvé que toute intégrale de la forme

$$Y_t := \delta(u_\cdot 1_{[0,t]\times\mathbb{R}}(\cdot)), \qquad t \in [0,1]$$

peut s'écrire comme une intégrale de Itô-Skorohod dans le sens de [111].

Proposition 3. *Soit $u \in L^{k,2}$, avec $k \ge 3$. Alors il existe un processus unique $v \in L^{k-2,2}$ tel que pour tout $0 < t \le 1$*

$$Y_t := \delta(u_\cdot 1_{[0,t]\times\mathbb{R}}(\cdot)) = \int\int_{(0,t]\times\mathbb{R}} {}^{(p,t)}(v_{s,x})\ M(ds,dx).$$

De plus $v_{s,x} = D_{s,x}Y_s \quad \mu \otimes P$ presque partout sur $[0,1]\times\mathbb{R}\times\Omega$.

En utilisant ce dernier résultat, nous avons pu établir une formule d'Itô pour des intégrales anticipés sur l'espace de Lévy.

Proposition 4 (Formule d'Itô). *Supposons que v est un processus appartenant à $L^2([0,1]\times\mathbb{R}\times\Omega, \mu\otimes P)$. Définissons*

$$Y_t = \int\int_{(0,t]\times\mathbb{R}} E\left(v_{s,x}/F^X_{[s,t]^c}\right) M(ds,dx)$$

et soit f une fonction de classe C^2. Alors

$$f(Y_t) = f(0) + \int\int_{(0,t]\times\mathbb{R}} f'(Y_t^{s^-})^{(p,t)}(D_{s,x}Y_s)\ M(ds,dx)$$

$$+\frac{1}{2}\int\int_{(0,t]\times\mathbb{R}} f''(Y_t^{s^-})({}^{(p,t)}(D_{s,0}Y_s))^2\ ds$$

$$+\sum_{0<s\le t}\left(f(Y_t^s) - f(Y_t^{s^-}) - f'(Y_t^{s^-})(Y_t^s - Y_t^{s^-})\right)$$

où $Y_t^s := \int\int_{(0,s]\times\mathbb{R}} E\left(v_{s,x}/F^X_{[s,t]^c}\right) M(ds,dx)$ et $Y_t^{s^-} = \lim_{r\to s^-} Y_t^r$ pour tout $0 < s \le t$.

Ce chapitre fait l'objet d'une publication [40] en collaboration avec C. Tudor.

Chapitre 6 : Classe des processus qui sont infiniment divisible en temps

Dans ce chapitre, nous donnons un lien entre processus stochastique, infiniment divisible en temps (IDT), et processus de Lévy. Nous étudions la connexion entre autosimilarité et stabilité stricte pour les processus IDT. Nous considérons aussi une subordination d'un processus de Lévy à travers un processus IDT croissant. Enfin, nous introduisons une notion: celle des processus stochastiques multiparamètre IDT, extension naturelle de celle introduite par Mansuy [72].

Les processus IDT ont été introduits par Mansuy [72] comme une généralisations de processus de Lévy. La motivation fut un travail de Barndorff-Nielsen et Thorbjørnsen [8]. Il s'agit de processus $X = (X_t, t \geq 0)$ à valeurs dans $\mathbb{R}^d$ qui ont la propriété dite d'infiniment divisible en temps : Pour tout $n \in \mathbb{N}^*$, la loi de $(X_{nt}, t \geq 0)$ est la loi de

$$(X_t^{(1)} + ... + X_t^{(n)}, t \geq 0),$$

où $X^{(1)}, \ldots, X^{(n)}$ sont des copies indépendantes de X.
Plusieurs propriétés des processus IDT ont été étudiées dans [72], concernant par exemple la caractéristique des processus IDT gaussien et leur autodécomposabilité temporale.

Le but du chapitre 6 est d'étendre certains résultats sur les processus de Lévy étudiés dans [7], [35] et [73] au cas des processus IDT.
Le théorème suivant énonce une condition nécessaire et suffisante pour qu'un processus IDT soit un processus de Lévy: l'hypothèse d'accroissements indépendants.

Théorème 7. *Si $X = (X_t, t \geq 0)$ est un IDT continu en probabilité à accroissements indépendants, alors X est un processus de Lévy.*

Soit $0 < \alpha \leq 2$. Une mesure de probabilité infiniment divisible μ est appelée strictement $\alpha-$stable si, pour tout $a > 0$

$$\hat{\mu}(\theta)^a = \hat{\mu}(a^{1/\alpha}\theta), \ \forall \ \theta \in \mathbb{R}^d$$

où

$$\hat{\mu}(\theta) = \int_{\mathbb{R}^d} e^{i<\theta,z>} \mu(dz).$$

On dit qu'un processus $X = (X_t, t \geq 0)$ est strictement $\alpha-$stable si toute loi finie-dimensionnelle de X est strictement $\alpha-$stable.
Nous étudions maintenant le lien entre trois notions de processus : processus autosimilaire, processus strictement stable et processus IDT.

Théorème 8. *Soit $0 < \alpha \leq 2$. Si $X = (X_t, t \geq 0)$ est un processus stochastique continu en probabilité, alors lorsque nous combinons deux propriétés parmi les trois suivantes nous obtenons la troisième.*

♠ *X est strictement α-stable.*

♣ *X est $(\frac{1}{\alpha})$-autosimilaire.*

♦ *X est IDT.*

Dans le cas où $X = (X_t, t \geq 0)$ est un processus de Lévy, Embrechts et Maejima [35] ont montré l'équivalence entre les deux propriétés (♠) et (♣). Nous donnons un exemple qui prouve que le théorème 8 n'est pas vrai en général si nous remplaçons la propriété (♦) par: X est un processus de Lévy. Soit S_α une variable aléatoire strictement α-stable. Le processus X défini par

$$X_t = t^{1/\alpha} S_\alpha, \ t \geq 0,$$

est un processus $(1/\alpha)$-autosimilaire, α-stable et IDT mais n'est pas de Lévy.
Nous étudions également la connexion entre la semi-autosimilarité et la semi-stabilité stricte pour les processus IDT (voir la sous-section 6.3.2). Cette connexion a été démontrée par Sato [105] pour les processus de Lévy.

La subordination est une transformation d'un processus stochastique en un nouveau processus stochastique, à travers changement du temps aléatoire par un processus de Lévy croissant (subordinateur) indépendant du processus original. Nous énonçons notre résultat obtenu dans ce cadre.

Théorème 9. *Soit X un processus de Lévy à valeurs dans $\mathbb{R}^d$ et ξ un processus IDT croissant continu en probabilité tels que X et ξ sont indépendants. Alors $(Z_t := X_{\xi_t} : t \geq 0)$ est un processus IDT.*

Dans la section 6.4 nous introduisons la notion de processus multiparamètre infiniment divisible en temps. Puis nous caractérisons les processus gaussien multiparamètre qui sont IDT. Ensuite, plusieurs prpriétés ont été étudiées comme dans le cas des processus IDT avec un seul paramètre.
Ce chapitre fait l'objet d'une publication [38] en collaboration avec Y. Ouknine.

Les six chapitres de cette thèse, qui correspondent chacun à un article, publié ou soumis dans une revue scientifique à comité de lecture, sont indépendants les uns des autres. Il en est de même pour les notations utilisées, qui peuvent varier d'un chapitre à l'autre.

Part I

MALLIAVIN CALCULUS AND LOCAL TIME ON GAUSSIAN SPACE

Chapter 1

Multidimensional bifractional Brownian motion: Itô and Tanaka formulas

Using the Malliavin calculus with respect to Gaussian processes and the multiple stochastic integrals we derive Itô's and Tanaka's formulas for the d-dimensional bifractional Brownian motion.

1.1 Introduction

The stochastic calculus with respect to the fractional Brownian motion (fBm) has now a long history. Since the nineties, many authors used different approaches to develop a stochastic integration theory with respect to this process. We refer, among of course many others, to [1], [27], [33] or [46]. The reason for this tremendous interest in the stochastic analysis of the fBm comes from its large amount of applications in practical phenomena such as telecommunications, hydrology or economics.

Nevertheless, even fBm has its limits in modeling certain phenomena. Therefore, several authors introduced recently some generalizations of the fBm which are supposed to fit better in concrete situations. For example, we mention the multifractional Brownian motion (see e.g. [4]), the subfractional Brownian motion (see e.g. [16]) or the multiscale fractional Brownian motion (see [5]).

Here our main interest consists in the study of the *bifractional Brownian motion (bifBm).* The bifBm has been introduced by Houdré and Villa in [47] and a stochastic analysis for it can be found in [103]. Other papers treated different aspects of this stochastic process, like sample paths properties, extension of the parameters or statistical applications (see [17], [12], [113] or [25]). Recall that the bifBm $B^{H,K}$ is a centered Gaussian process, starting from zero, with covariance function

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right) \tag{1.1}$$

where the parameters H, K are such that $H \in (0,1)$ and $K \in (0,1]$. In the case $K = 1$ we retrieve the fractional Brownian motion while the case $K = 1$ and $H = \frac{1}{2}$ corresponds to the standard Brownian motion.

The process $B^{H,K}$ is HK-selfsimilar but it has no stationary increments. It has Hölder continuous paths of order $\delta < HK$ and its paths are not differentiable. An interesting property of it is the fact that its quadratic variation in the case $2HK = 1$ is similar to that of the standard Brownian motion, i.e. $[B^{H,K}]_t = cst. \times t$ and therefore especially this case ($2HK = 1$) is very interesting from the stochastic calculus point of view.

In this paper, our purpose is to study multidimensional bifractional Brownian motion and to prove Itô and Tanaka formulas. We start with the one dimensional bifBm and we first derive Itô and Tanaka formulas for it when $2HK \geq 1$. We mention that the Itô formula has been already proved by [61] but here we propose an alternative proof based on the Taylor expansion which appears to be also useful in the multidimensional settings. The Tanaka formula is obtained from the Itô formula by a limit argument and it involves the so-called *weighted local time* extending the result in [24]. In the multidimensional case we first derive an Itô formula for $2HK > 1$ and we extend it to Tanaka by following an idea by Uemura [114], [115]; that is, since $|x|$ is twice the kernel of the one-dimensional Newtonian potential, i.e. $\frac{1}{2}\Delta|x|$ is equal to the delta Dirac function $\delta(x)$, we will chose the function $U(z), z \in \mathbb{R}^d$ which is twice of the kernel of d-dimensional Newtonian (or logarithmic if $d = 2$) potential to replace $|x|$ in the d-dimensional case. See the last section for the definition of the function U. Our method is based on the Wiener-Itô chaotic expansion into multiple stochastic integrals following ideas from [54] or [34]. The multidimensional Tanaka formula also involves a generalized local time. We note that the terms appearing in our Tanaka formula when $d \geq 2$ are not random variables and they are understood as distributions in the Watanabe spaces.

1.2 Preliminaries: Deterministic spaces associated and Malliavin calculus

Let $\left(B_t^{H,K}, t \in [0,T]\right)$ be a bifractional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$.

Being a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $B^{H,K}$. We refer to [1], [81] for a complete description of stochastic calculus with respect to Gaussian processes. Here we recall only the basic elements of this theory.

The basic ingredient is the canonical Hilbert space $\mathcal{H}$ associated to the bifractional Brownian motion. This space is defined as the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

The application $\varphi \in \mathcal{E} \to B^{H,K}(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{H,K}$ and it can be extended to $\mathcal{H}$.

Let us denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^{H,K}(\varphi_1), \dots, B^{H,K}(\varphi_n))$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of a functional F as above is given by

$$D^{B^{H,K}} F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^{H,K}(\varphi_1), \dots, B^{H,K}(\varphi_n))\varphi_i$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}$ $(m \geq 1)$ of $\mathcal{S}$ with respect to the norm

$$\|F\|_{m,2}^2 := E\,|F|^2 + E\|D^{B^{H,K}}F\|_{\mathcal{H}}^2 + \dots + E\|D^{B^{H,K},m}F\|_{\mathcal{H}^{\hat{\otimes} m}}^2$$

where $\mathcal{H}^{\hat{\otimes} m}$ denotes the m fold symmetric tensor product and the mth derivative $D^{B^{H,K},m}$ is defined by iteration.

The divergence integral $\delta^{B^{H,K}}$ is the adjoint operator of $D^{B^{H,K}}$. Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $(Dom(\delta^{B^{H,K}}))$ if

$$E\left|\langle D^{B^{H,K}}F, u\rangle_{\mathcal{H}}\right| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta^{B^{H,K}}(u)$ is given by the duality relationship

$$E(F\delta^{B^{H,K}}(u)) = E\langle D^{B^{H,K}}F, u\rangle_{\mathcal{H}}$$

for any $F \in \mathbb{D}^{1,2}$. It holds that

$$E\delta^{B^{H,K}}(u)^2 = E\|u\|_{\mathcal{H}}^2 + E\langle D^{B^{H,K}}u, (D^{B^{H,K}}u)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \tag{1.2}$$

where $(D^{B^{H,K}}u)^*$ is the adjoint of $D^{B^{H,K}}u$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

Sometimes working with the space $\mathcal{H}$ is not convenient; once, because this space may contain also distributions (as, e.g. in the case $K = 1$, see [95]) and twice, because the norm in this space is not always tractable. We will use the subspace $|\mathcal{H}|$ of $\mathcal{H}$ which is defined as the set of measurable function f on $[0, T]$ with

$$\|f\|_{|\mathcal{H}|}^2 := \int_0^T \int_0^T |f(u)|\,|f(v)| \left|\frac{\partial^2 R}{\partial u \partial v}(u, v)\right| dudv < \infty. \tag{1.3}$$

It follows actually from [61] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. In fact,

$$L^2([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

and

$$E\delta^{B^{H,K}}(u)^2 \leq E\|u\|_{|\mathcal{H}|}^2 + E\|D^{B^{H,K}}u\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 \tag{1.4}$$

where, if $\varphi : [0, T]^2 \to \mathbb{R}$

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 = \int_{[0,T]^4} |\varphi(u, v)|\,|\varphi(u', v')| \left|\frac{\partial^2 R}{\partial u \partial u'}(u, u')\frac{\partial^2 R}{\partial v \partial v'}(v, v')\right| dudvdu'dv'. \tag{1.5}$$

We will use the following formulas of the Malliavin calculus: the integration by parts

$$F\delta^{B^{H,K}}(u) = \delta^{B^{H,K}}(Fu) + \langle D^{B^{H,K}}F, u\rangle_{\mathcal{H}} \tag{1.6}$$

for any $u \in Dom(\delta^{B^{H,K}})$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega; \mathcal{H})$; and the chain rule

$$D^{B^{H,K}}\varphi(F) = \sum_{i=1}^{n} \partial_i\varphi(F) D^{B^{H,K}} F^i$$

if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and $F = (F^1, \ldots, F^m)$ is a random vector with components in $\mathbb{D}^{1,2}$.

By the duality between $D^{B^{H,K}}$ and $\delta^{B^{H,K}}$ we obtain the following result for the convergence of divergence integrals: if $u_n \in Dom(\delta^{B^{H,K}})$ for every n, $u_n \underset{n\to\infty}{\to} u$ in $L^2(\Omega; \mathcal{H})$ and $\delta^{B^{H,K}}(u_n) \underset{n\to\infty}{\to} G \in L^2(\Omega)$ in $L^1(\Omega)$ then

$$u \in Dom(\delta^{B^{H,K}}) \text{ and } \delta^{B^{H,K}}(u) = G. \tag{1.7}$$

It is also possible to introduce multiple integrals $I_n(f_n)$, $f \in \mathcal{H}^{\otimes n}$ with respect to $B^{H,K}$. Let

$$F = \sum_{n\geq 0} I_n(f_n) \tag{1.8}$$

where for every $n \geq 0$, $f_n \in \mathcal{H}^{\otimes n}$ are symmetric functions. Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} nI_n(f_n)$$

if F is given by (1.8).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I - L)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. In this way, a random variable F as in (1.8) belongs to $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geq 0}(1+n)^{\alpha}\|I_n(f_n)\|^2_{L^2(\Omega)} < \infty.$$

Note that the Malliavin derivative operator acts on multiple integral as follows

$$D_t^{B^{H,K}}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)), \qquad t \in [0, T].$$

The operator $D^{B^{H,K}}$ is continuous from $\mathbb{D}^{\alpha-1,p}$ into $\mathbb{D}^{\alpha,p}(\mathcal{H})$. The adjoint of $D^{B^{H,K}}$ is denoted by $\delta^{B^{H,K}}$ and is called the divergence (or Skorohod) integral. It is a continuous operator from

$\mathbb{D}^{\alpha,p}(\mathcal{H})$ into $\mathbb{D}^{\alpha,p}$. For adapted integrands, the divergence integral coincides to the classical Itô integral. We will use the notation

$$\delta^{B^{H,K}}(u) = \int_0^T u_s \delta B_s^{H,K}.$$

Recall that if u is a stochastic process having the chaotic decomposition

$$u_s = \sum_{n\geq 0} I_n(f_n(\cdot, s))$$

where $f_n(\cdot, s) \in \mathcal{H}^{\otimes n}$ for every s, and it is symmetric in the first n variables, then its Skorohod integral is given by

$$\int_0^T u_s dB_s^{H,K} = \sum_{n\geq 0} I_{n+1}(\tilde{f}_n)$$

where $\tilde{f}_n$ denotes the symmetrization of f_n with respect to all $n+1$ variables.

1.3 Tanaka formula for unidimensional bifractional Brownian motion

This paragraph is consecrated to the proof of Itô formula and Tanaka formula for the one-dimensional bifractional Brownian motion with $2HK \geq 1$. Note that the Itô formula has been already proved in [61]; here we propose a different approach based on the Taylor expansion which will be also used in the multidimensional settings.

We start by the following technical lemma.

Lemma 1. *Let us consider the following function on* $[1, \infty)$

$$h(y) = y^{2HK} + (y-1)^{2HK} - \frac{2}{2^K}\left(y^{2H} + (y-1)^{2H}\right)^K.$$

where $H \in (0,1)$ *and* $K \in (0,1)$. *Then,*

$$h(y) \text{ converges to } 0 \text{ as } y \text{ goes to } \infty. \tag{1.9}$$

Moreover if $2HK = 1$ *we obtain that*

$$\lim_{y\to+\infty} yh(y) = \frac{1}{4}(1-2H). \tag{1.10}$$

Proof: Let $y = \frac{1}{\varepsilon}$, hence

$$h(y) = h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left[1 + (1-\varepsilon)^{2HK} - \frac{2}{2^K}\left(1 + (1-\varepsilon)^{2H}\right)^K\right].$$

Using Taylor's expansion, as ε close to 0, we obtain

$$h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left(H^2K(K-1)\varepsilon^2 + o(\varepsilon^2)\right). \tag{1.11}$$

Thus

$$\lim_{y\to+\infty} h(y) = \lim_{\varepsilon\to 0} h(1/\varepsilon) = 0.$$

For the case $2HK = 1$ we replace in (1.11), we have

$$\frac{1}{\varepsilon}h\left(\frac{1}{\varepsilon}\right) = \frac{1}{4}(1-2H) + o(1).$$

Thus (1.10) is satisfied. Which completes the proof. ■

Theorem 1. *Let f be a function of class C^2 on $\mathbb{R}$ such that*

$$\max\{|f(x)|, |f'(x)|, |f''(x)|\} \leq ce^{\beta x^2} \tag{1.12}$$

where c and β are positive constants such that $\beta < \frac{1}{4T^{2HK}}$. Suppose that $2HK \geq 1$. Then $f'(B^{H,K}) \in Dom(\delta^{B^{H,K}})$ and for every $t \in [0,T]$

$$f\left(B_t^{H,K}\right) = f(0) + \int_0^t f'\left(B_s^{H,K}\right)\delta B_s^{H,K} + HK\int_0^t f''\left(B_s^{H,K}\right)s^{2HK-1}ds. \tag{1.13}$$

Proof: We first prove the case $2HK > 1$. It follows from (1.12) (as in e.g. [1]) that $f'(B^{H,K}) \in L^2(\Omega; |\mathcal{H}|)$. Let us fix $t \in [0,T]$ and let be $\pi^n := \{t_j^n = \frac{jt}{n}; j = 0, ..., n\}$ a partition of $[0,t]$. Using Taylor expansion, we have

$$\begin{aligned} f\left(B_t^{H,K}\right) &= f(0) + \sum_{j=1}^n f'\left(B_{t_{j-1}^n}^{H,K}\right)\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right) + \frac{1}{2}\sum_{j=1}^n f''\left(\overline{B}_j^{H,K}\right)\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^2 \\ &:= f(0) + I^n + J^n. \end{aligned} \tag{1.14}$$

with $\overline{B}_j^{H,K} = B_{t_{j-1}^n}^{H,K} + \theta_j\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)$ where θ_j is a r.v in $(0,1)$.
The growth condition (1.12) implies

$$E\left(\sup_{0\leq s\leq T}|f\left(B_s^{H,K}\right)|^p\right) \leq c^pE\left(e^{p\sup_{0\leq s\leq T}|\left(B_s^{H,K}\right)|^2}\right) < \infty \tag{1.15}$$

for any $p < \frac{1}{2\beta T^{2HK}}$. In particular for $p = 2$. The same property holds for f' and f''. Combining this with the fact that $B^{H,K}$ is a quasi-helix (see [103]), we can bound the term J^n as follows:

$$\begin{aligned} E|J^n| &\leq \frac{1}{2}E\left(\sup_{0\leq s\leq T}|f''\left(B_s^{H,K}\right)|\sum_{j=1}^n\left(B_{t_j^n}^{H,K}-B_{t_{j-1}^n}^{H,K}\right)^2\right) \\ &\leq \frac{1}{2}\left\|\sup_{0\leq s\leq T}|f''\left(B_s^{H,K}\right)|\right\|_{L^2(\Omega)}\left\|\sum_{j=1}^n\left(B_{t_j^n}^{H,K}-B_{t_{j-1}^n}^{H,K}\right)^2\right\|_{L^2(\Omega)} \\ &\leq C(H,K)\sum_{j=1}^n|t_j^n-t_{j-1}^n|^{2HK} \\ &\leq C(H,K)\frac{T^{2HK}}{n^{2HK-1}}\underset{n\to\infty}{\longrightarrow}0 \end{aligned}$$

where $C(H,K)$ a constant depends on H and K. Then

$$J^n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^1(\Omega). \tag{1.16}$$

On the other hand, we apply (1.6) we get

$$\begin{aligned} I^n &= \sum_{j=1}^n f'\left(B_{t_{j-1}^n}^{H,K}\right)\delta^{B^{H,K}}(1_{(t_{j-1}^n,t_j^n]}) \\ &= \delta^{B^{H,K}}\left(\sum_{j=1}^n f'\left(B_{t_{j-1}^n}^{H,K}\right)1_{(t_{j-1}^n,t_j^n]}(.)\right)+\sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)\langle 1_{(0,t_{j-1}^n]},1_{(t_{j-1}^n,t_j^n]}\rangle_{\mathcal{H}} \\ &= I_1^n+I_2^n. \end{aligned}$$

Next

$$\begin{aligned} I_2^n &= \sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)\left(R(t_{j-1}^n,t_j^n)-R(t_{j-1}^n,t_{j-1}^n)\right) \\ &= \sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)\left(\frac{1}{2^K}\left(\left((t_j^n)^{2H}+(t_{j-1}^n)^{2H}\right)^K-(t_j^n-t_{j-1}^n)^{2HK}\right)-(t_{j-1}^n)^{2HK}\right) \\ &:= \sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)b(j). \end{aligned}$$

We denote by

$$A_t := HK\int_0^t s^{2HK-1}ds=\frac{1}{2}t^{2HK}.$$

To prove that I_2^n converges to $HK\int_0^t f''\left(B_s^{H,K}\right)s^{2HK-1}ds$ in $L^1(\Omega)$ as $n\to\infty$, it suffices to show that

$$C_n := E\left|I_2^n-\sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)\left(A_{t_j^n}-A_{t_{j-1}^n}\right)\right|\underset{n\to\infty}{\longrightarrow}0.$$

By Minkowski inequality, we have

$$\begin{aligned} C_n &\leq E\left(\sup_{0\leq s\leq T}|f\left(B_s^{H,K}\right)|\right)\sum_{j=1}^{n}\left|b(j)-\frac{1}{2}((t_j^n)^{2HK}-(t_{j-1}^n)^{2HK})\right| \\ &\leq C(H,K,T)\left[\frac{1}{n^{2HK}}\sum_{j=1}^{n}|h(j)|+\frac{2}{2^K}\frac{1}{n^{2HK-1}}\right] \\ &= C(H,K,T)\left[C_n^1+C_n^2\right] \end{aligned}$$

where $C(H,K,T)$ is a generic constant depends on H, K and T.
Since $2HK>1$ then $C_n^2:=\frac{2}{2^K}\frac{1}{n^{2HK-1}}\underset{n\to\infty}{\longrightarrow}0$. According to (1.9), we obtain

$$C_n^1:=\frac{1}{n^{2HK}}\sum_{j=1}^{n}h(j)\leq\frac{C}{n^{2HK-1}}\underset{n\to\infty}{\longrightarrow}0.$$

Thus

$$I_2^n\underset{n\to\infty}{\longrightarrow}HK\int_0^t f''\left(B_s^{H,K}\right)s^{2HK-1}ds \text{ in } L^1(\Omega).$$

We show now that

$$u_\cdot^n:=\sum_{j=1}^{n}f'\left(B_{t_{j-1}^n}^{H,K}\right)1_{(t_{j-1}^n,t_j^n]}(.)\underset{n\to\infty}{\longrightarrow}u_\cdot:=f'\left(B_\cdot^{H,K}\right)1_{(0,t]}(.) \text{ in } L^2(\Omega;\mathcal{H}). \tag{1.17}$$

Indeed, using (1.12) and the continuity of the process $f'\left(B^{H,K}\right)$, we obtain

$$\begin{aligned} E\|u^n-u\|_{|\mathcal{H}|}^2 &= E\|\sum_{j=1}^{n}\left[f'\left(B_{t_{j-1}^n}^{H,K}\right)-f'\left(B_\cdot^{H,K}\right)\right]1_{(t_{j-1}^n,t_j^n]}(.)\|_{|\mathcal{H}|}^2 \\ = & E\sum_{j,l=1}^{n}\int_{t_{j-1}^n}^{t_j^n}\int_{t_{l-1}}^{t_l}\left|f'\left(B_{t_{j-1}^n}^{H,K}\right)-f'\left(B_r^{H,K}\right)\right|\left|f'\left(B_{t_{l-1}}^{H,K}\right)-f'\left(B_s^{H,K}\right)\right|\left|\frac{\partial^2R}{\partial r\partial s}(r,s)\right|drds \\ \leq & E\left(\sup_{|r-s|\leq\frac{t}{n}}\left|f'\left(B_r^{H,K}\right)-f'\left(B_s^{H,K}\right)\right|\right)^2\sum_{j,l=1}^{n}\int_{t_{j-1}^n}^{t_j^n}\int_{t_{l-1}}^{t_l}\left|\frac{\partial^2R}{\partial r\partial s}(r,s)\right|drds \\ = & E\left(\sup_{|r-s|\leq\frac{t}{n}}\left|f'\left(B_r^{H,K}\right)-f'\left(B_s^{H,K}\right)\right|\right)^2C(T)\to_{n\to\infty}0. \end{aligned}$$

The above steps prove that I_1^n converges in $L^1(\Omega)$ to $f(B_t^{H,K})-f(0)-HK\int_0^t f"(B_s^{H,K})s^{2HK-1}ds$. By combining this and (1.17), by property (1.7) we deduce $f'(B^{H,K})\in Dom(\delta^{B^{H,K}})$ and

$$I_1^n \text{ converges to } \delta^{B^{H,K}}\left(f'\left(B_\cdot^{H,K}\right)1_{(0,t]}(.)\right) \text{ in } L^2(\Omega).$$

Therefore (1.13) is established. ■

The proof of the case $2HK = 1$ is based on a preliminary result concerning the quadratic variation of the bifractional Brownian motion. It was proved in [103] using the stochastic calculus via regularization.

Lemma 2. *Suppose that* $2HK = 1$*, then*

$$V_t^n := \sum_{j=1}^{n} \left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K} \right)^2 \underset{n\to\infty}{\longrightarrow} \frac{1}{2^{k-1}} t \quad in\ L^2(\Omega).$$

Proof: A straightforward calculation shows that,

$$EV_t^n = \frac{t}{n} \sum_{j=1}^{n} h(j) + \frac{t}{2^{k-1}} \underset{n\to\infty}{\longrightarrow} \frac{t}{2^{k-1}}.$$

To obtain the conclusion it suffices to show that

$$\lim_{n\to\infty} E(V_t^n)^2 = (\frac{t}{2^{k-1}})^2.$$

In fact we have,

$$E(V_t^n)^2 = \sum_{i,j=1}^{n} E\left((B_{t_i^n}^{H,K} - B_{t_{i-1}^n}^{H,K})(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}) \right)^2$$

Denote by

$$\mu_n(i,j) = E\left((B_{t_i^n}^{H,K} - B_{t_{i-1}^n}^{H,K})(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}) \right)^2$$

It follows by linear regression that

$$\mu_n(i,j) = E\left(N_1^2 \left| \theta_n(i,j) N_1 + \sqrt{\delta_n(i,j) - (\theta_n(i,j))^2} N_2 \right|^2 \right)$$

where N_1 and N_2 two independent normal random variables,

$$\begin{aligned} \theta_n(i,j) &:= E\left((B_{t_i^n}^{H,K} - B_{t_{i-1}^n}^{H,K})(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}) \right) \\ &= \frac{t}{2^K n} \left[(i^{2H} + j^{2H})^K - 2|j-i| - (i^{2H} + (j-1)^{2H})^K + |j-i-1| \right. \\ &\quad \left. -((i-1)^{2H} + j^{2H})^K + |j-i+1| + ((i-1)^{2H} + (j-1)^{2H})^K \right] \end{aligned}$$

and

$$\delta_n(i,j) := E\left(B_{t_i^n}^{H,K} - B_{t_{i-1}^n}^{H,K} \right)^2 E\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K} \right)^2.$$

Hence

$$\mu_n(i,j) = 2(\theta_n(i,j))^2 + \delta_n(i,j).$$

For $1 \leq i < j$, we define a function $f_j : (1, \infty) \to \mathbb{R}$, by

$$\begin{aligned} f_j(x) &= ((x-1)^{2H} + j^{2H})^K - ((x-1)^{2H} + (j-1)^{2H})^K \\ &\quad -(x^{2H} + j^{2H})^K + (x^{2H} + (j-1)^{2H})^K. \end{aligned}$$

We compute

$$\begin{aligned} f_j'(x) &= \left(\frac{(x-1)^{2H} + j^{2H}}{(x-1)^{2H}}\right)^{K-1} - \left(\frac{(x-1)^{2H} + (j-1)^{2H}}{(x-1)^{2H}}\right)^{K-1} \\ &\quad - \left(\frac{x^{2H} + j^{2H}}{x^{2H}}\right)^{K-1} + \left(\frac{x^{2H} + (j-1)^{2H}}{x^{2H}}\right)^{K-1} \\ &:= g(x-1) - g(x) \geq 0 \end{aligned}$$

Hence f_j is increasing and positive, since the function

$$g(x) = \left(1 + \frac{j^{2H}}{x^{2H}}\right)^{K-1} - \left(1 + \frac{(j-1)^{2H}}{x^{2H}}\right)^{K-1}$$

is decreasing on $(1, \infty)$. This implies that for every $1 \leq i < j$

$$|\theta_n(i,j)| = \frac{t}{2^K n} f_j(i) \leq \frac{t}{2^K n} f_j(j) \leq \frac{t}{n} |h(j)|$$

and $|\theta_n(i,i)| = \frac{t}{n}|h(i) + 2|$ for any $i \geq 1$.
Thus

$$\sum_{i,j=1}^{n} \theta_n(i,j)^2 \leq \frac{2t^2}{n^2} \sum_{\substack{i<j \\ i,j=1}}^{n} h(j)^2 + \frac{t^2}{n^2} \sum_{i=1}^{n} (h(i)+2)^2.$$

Combining this with (1.10), we obtain that $\sum_{i,j=1}^{n} \theta_n(i,j)^2$ converges to 0 as $n \to \infty$.
On the other hand, by (1.10)

$$\sum_{i,j=1}^{n} \delta_n(i,j) = \frac{t^2}{n^2} \sum_{i,j=1}^{n} \left(h(i) + \frac{1}{2^{K-1}}\right)\left(h(j) + \frac{1}{2^{K-1}}\right) \underset{n\to\infty}{\longrightarrow} \left(\frac{t}{2^{K-1}}\right)^2.$$

Consequently, $E(V_t^n)^2$ converges to $\left(\frac{t}{2^{K-1}}\right)^2$ as $n \to \infty$, and the conclusion follows. ∎

Proof: [Proof of the Theorem 1 in the case $2HK = 1$.] In this case we shall prove that

$$I_1^n \underset{n\to\infty}{\longrightarrow} \int_0^t f'\left(B_s^{H,K}\right) \delta B_s^{H,K} \text{ in } L^2(\Omega), \tag{1.18}$$

$$I_2^n \underset{n\to\infty}{\longrightarrow} \left(\frac{1}{2} - \frac{1}{2^K}\right) \int_0^t f''\left(B_s^{H,K}\right) ds \text{ in } L^2(\Omega), \tag{1.19}$$

and

$$J^n \underset{n\to\infty}{\longrightarrow} \frac{1}{2^K}\int_0^t f''\left(B_s^{H,K}\right) ds \text{ in } L^1(\Omega). \tag{1.20}$$

To prove (1.19), it is enough to establish that

$$E_n := \left(E\left|I_2^n - \left(\frac{1}{2}-\frac{1}{2^K}\right)\sum_{j=1}^n f''\left(B_{t_{j-1}^n}^{H,K}\right)\left(t_j^n - t_{j-1}^n\right)\right|^2\right)^{1/2} \underset{n\to\infty}{\longrightarrow} 0.$$

According to (1.12) and (1.10), we obtain

$$\begin{aligned}
E_n &\leq \left(E\left(\sup_{0\leq s\leq T}\left|f\left(B_s^{H,K}\right)\right|\right)^2\right)^{1/2}\sum_{j=1}^n\left|\frac{1}{2^K}((t_j^n)^{2H}+(t_{j-1}^n)^{2H})^K - \frac{1}{2}(t_j^n + t_{j-1}^n)\right| \\
&\leq \frac{C(H,K)}{2n}\sum_{j=1}^n\left|2j-1-\frac{2}{2^K}(j^{2H}+(j-1)^{2H})^K\right| \\
&= \frac{C(H,K)}{2n}\sum_{j=1}^n h(j) \leq \frac{C(H,K)}{2n}\sum_{j=1}^n\frac{1}{j}\underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Suppose that $n\geq m$, and for any $j=1,...,n$ let us denote by $t_j^{m(n)}$ the point of the mth partition that is closer to t_j^n from the left. Then we obtain

$$\begin{aligned}
& E\left|J^n - \frac{1}{2^K}\int_0^t f''\left(\overline{B}_s^{H,K}\right) ds\right| \\
\leq\ & \frac{1}{2}E\left|\sum_{j=1}^n\left(f''\left(\overline{B}_j^{H,K}\right) - f''\left(B_{t_{j-1}^{m(n)}}^{H,K}\right)\right)\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^2\right| \\
+\ & \frac{1}{2}E\left|\sum_{k=1}^m f''\left(B_{t_{k-1}^m}^{H,K}\right)\sum_{j:t_{k-1}^m\leq t_{j-1}^n<t_k^m}\left(\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^2 - \frac{t_j^n - t_{j-1}^n}{2^{K-1}}\right)\right| \\
& +\frac{1}{2^K}E\left|\sum_{k=1}^m\int_{t_{k-1}}^{t_k}\left(f''\left(B_{t_{k-1}^m}^{H,K}\right) - f''\left(B_s^{H,K}\right)\right) ds\right| \\
\leq\ & \frac{1}{2}E\left(\sup_{|r-s|\leq\frac{t}{n}}\left|f''\left(B_r^{H,K}\right) - f''\left(B_s^{H,K}\right)\right|\sum_{j=1}^n\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^2\right) \\
+\ & \frac{1}{2}E\left|\sum_{k=1}^m f''\left(B_{t_{k-1}^m}^{H,K}\right)\sum_{j:t_{k-1}^m\leq t_{j-1}^n{}^n<t_k^m}\left(\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^2 - \frac{t_j^n - t_{j-1}^n}{2^{K-1}}\right)\right| \\
& +\frac{t}{2^K}E\left(\sup_{|r-s|\leq\frac{t}{m}}\left|f''\left(B_r^{H,K}\right) - f''\left(B_s^{H,K}\right)\right|\right).
\end{aligned}$$

By using Lemma 2, let us tender n to infinity and m to infinity. We deduce that (1.20) holds. The rest of the proof is same as in the case $2HK>1$. ■

Remark 1. *The "pathwise" Itô formula for the bifBm has been proved in [103] for any $H \in [0,1)$ and $K \in (0,1]$ by using the so-called Newton-Cotes integral introduced in [46]. One can note that the Skorohod integral $\int_0^t f'(B_s^{H,K})\delta B_s^{H,K}$ appearing in Theorem 1 is equal to the pathwise integral $\int_0^t f'(B_s^{H,K})dB_s^{H,K}$ minus the "trace" $HK\int_0^t f''(B_s^{H,K})s^{2HK-1}ds$.*

Let us regard now the Tanaka formula. As in the case of the standard fractional Brownian motion, it will involve the so-called *weighted local time* L_t^x ($x \in \mathbb{R}$, $t \in [0,T]$) of $B^{H,K}$ defined as the density of the occupation measure

$$A \in \mathcal{B}(\mathbb{R}) \longrightarrow 2HK \int_0^t 1_A(B_s^{H,K})s^{2HK-1}ds.$$

Theorem 2. *Let $\left(B_t^{H,K}, t \in [0,T]\right)$ be a bifractional Brownian motion with $2HK \geq 1$. Then for every $x \in \mathbb{R}$ we have $sign(B^{H,K}. - x) \in Dom(\delta^{B^{H,K}})$ and for each $t \in [0,T]$, $x \in \mathbb{R}$ the following formula holds*

$$\left|B_t^{H,K} - x\right| = |x| + \int_0^t sign(B_s - x)\delta B_s^{H,K} + L_t^x. \tag{1.21}$$

Proof: We follow the original proof of [24]. Let $p_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{y^2}{2\varepsilon}}$ be the Gaussian kernel and put

$$F_\varepsilon'(z) = 2\int_{-\infty}^z p_\varepsilon(y)dy - 1,$$

and

$$F_\varepsilon(z) = \int_0^z F_\varepsilon'(y)dy.$$

By the Theorem 1 we have

$$\begin{aligned} F_\varepsilon\left(B_t^{H,K} - x\right) &= F_\varepsilon(-x) + \int_0^t F_\varepsilon'\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \\ &\quad + HK\int_0^t p_\varepsilon\left(B_s^{H,K} - x\right)s^{2HK-1}ds. \end{aligned} \tag{1.22}$$

Using (a slight adaptation of) Proposition 9 in [103] (or proposition 2 in [24]), one can prove that

$$L_t^x = \lim_{\varepsilon \to 0} 2HK \int_0^t p_\varepsilon(B_s^{H,K} - x)s^{2HK-1}ds \text{ in } L^2(\Omega) \tag{1.23}$$

and L_t^x admits the following chaotic representation into multiple stochastic integrals (here I_n represents the multiple integral with respect to the bifBm)

$$L_t^x = 2HK\sum_{n=0}^{\infty}\int_0^t \frac{p_{s^{2HK}}(x)}{s^{(n-2)HK+1}}H_n\left(\frac{x}{s^{HK}}\right)I_n(1_{[0,s]}^{\otimes n})ds \tag{1.24}$$

where H_n is the nth Hermite polynomial defined as

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}) \quad \text{for every } n \geq 1.$$

We have $F_\varepsilon(x) \to |x|$ as $\varepsilon \to 0$ and since $F_\varepsilon(x) \leq |x|$, then by Lebesgue's dominated convergence theorem we obtain that $F_\varepsilon(B_t^{H,K} - x)$ converges to $|B_t^{H,K} - x|$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

On the other hand, since $0 \leq F'_\varepsilon(x) \leq 1$ and $F'_\varepsilon(x) \to sign(x)$ as ε goes to 0 the Lebesgue's dominated convergence theorem in $L^2(\Omega\times[0,T]^2; P\otimes\left|\frac{\partial^2 R}{\partial u \partial v}(u,v)\right| dudv)$ implies that $F'_\varepsilon\left(B_\cdot^{H,K} - x\right)$ converges to $sign\left(B_\cdot^{H,K} - x\right)$ in $L^2(\Omega;\mathcal{H})$ as ε goes to 0 because

$$\begin{aligned} & E\|F'_\varepsilon\left(B_\cdot^{H,K} - x\right) - sign\left(B_\cdot^{H,K} - x\right)\|^2_{|\mathcal{H}|} \\ = \; & E\int_0^T\int_0^T \left|F'_\varepsilon\left(B_u^{H,K} - x\right) - sign\left(B_u^{H,K} - x\right)\right| \left|F'_\varepsilon\left(B_v^{H,K} - x\right) - sign\left(B_v^{H,K} - x\right)\right| \\ & \times \left|\frac{\partial^2 R}{\partial u \partial v}(u,v)\right| dudv. \end{aligned}$$

Consequently, from the above convergences and (1.7), we deduce as in the proof of Theorem 1 that $sign\left(B^{H,K} - x\right) \in Dom(\delta^{B^{H,K}})$ for every x and

$$\int_0^t F'_\varepsilon\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t sign\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \quad \text{in } L^2(\Omega).$$

Then the conclusion follows. ■

1.4 Tanaka formula for multidimensional bifractional Brownian motion

Given two vectors $H = (H_1,\dots,H_d) \in [0,1]^d$ and $K = (K_1,\dots,K_d) \in (0,1]^d$, we introduce the d-dimensional bifractional Brownian motion

$$B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$$

as a centered Gaussian vector whose component are independent one-dimensional bifractional Brownian motions.

We extend the Itô formula to the multidensional case.

Theorem 3. *Let $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ be a d-dimensional bifractional Brownian motion, and let f be a function of class $\mathrm{C}^2\left(\mathbb{R}^d,\mathbb{R}\right)$ such that for every $x \in \mathbb{R}^d$*

$$\max_{i,l=1}^d \left(|f(x)|, |\frac{\partial f}{\partial x_i}(x)|, |\frac{\partial^2 f}{\partial x_i \partial x_l}(x)|\right) \leq ce^{\beta|x|^2}, \tag{1.25}$$

where c and β are positive constants such that $\beta < \frac{1}{4T^{2(HK)^*}}$ where $(HK)^* = \max_{i=1}^d H^i K^i$. We assume that $2H_iK_i > 1$ for any $i = 1,...n$. Then for every i we have $\frac{\partial f}{\partial x_i}\left(B^{H_i,K_i}\right) \in Dom(\delta^{B_s^{H_i,K_i}})$ and for every $t \in [0,T]$

$$f\left(B_t^{H,K}\right) = f(0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}\left(B_s^{H_i,K_i}\right)\delta B_s^{H_i,K_i} + \sum_{i=1}^d H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(B_s^{H,K}\right) s^{2H_iK_i - 1} ds \tag{1.26}$$

Proof: Let us fix $t > 0$ and a partition $\{t_j^n = \frac{jt}{n}; j = 0, ..., n\}$ of $[0, t]$. As in above, the condition (1.25) implies that

$$E\left(\sup_{0\leq s\leq T} |f\left(B_s^{H,K}\right)|^2\right) \leq c^2 E\left(e^{2\sup_{0\leq s\leq T}|\left(B_s^{H,K}\right)|^2}\right) < \infty. \tag{1.27}$$

The same property holds for any $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_l}$ with $i, l = 1, ..., d$. Using Taylor expansion, we have

$$\begin{aligned} f\left(B_t^{H,K}\right) &= f(0) + \sum_{j=1}^n \sum_{i=1}^d \frac{\partial f}{\partial x_i}\left(B_{t_{j-1}^n}^{H,K}\right)\left(B_{t_j^n}^{H_i,K_i} - B_{t_{j-1}^n}^{H_i,K_i}\right) \\ &+ \frac{1}{2}\sum_{j=1}^n \sum_{i,l=1}^d \frac{\partial^2 f}{\partial x_i \partial x_l}\left(\overline{B}_j^{H,K}\right)\left(B_{t_j^n}^{H_i,K_i} - B_{t_{j-1}^n}^{H_i,K_i}\right)\left(B_{t_j^n}^{H_l,K_l} - B_{t_{j-1}^n}^{H_l,K_l}\right) \\ &:= f(0) + I^n + J^n \end{aligned}$$

where $\overline{B}_j^{H,K} = B_{t_{j-1}^n}^{H,K} + \theta_j\left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)$, and θ_j is a random variable in $(0, 1)$.
We show that J^n converges to 0 in $L^1(\Omega)$ as $n \to \infty$. Applying Hölder inequality and the property (1.27), we have

$$\begin{aligned} E|J^n| &\leq C(H,K)\sum_{j=1}^n \sum_{i,l=1}^d \left(E\left(B_{t_j^n}^{H_i,K_i} - B_{t_{j-1}^n}^{H_i,K_i}\right)^2\right)^{1/2}\left(E\left(B_{t_j^n}^{H_l,K_l} - B_{t_{j-1}^n}^{H_l,K_l}\right)^2\right)^{1/2} \\ &\leq C(H,K)\sum_{i,l=1}^d \frac{T^{2(HK)^*}}{n^{H_iK_i + H_lK_l - 1}} \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

(Note that if $2H_iK_i = 1$ for some i then the above sum does not converge to zero). According to (1.6), we get

$$\begin{aligned} I^n &= \sum_{j=1}^n \sum_{i=1}^d \frac{\partial f}{\partial x_i}\left(B_{t_{j-1}^n}^{H,K}\right)\left(\delta^{B^{H_i,K_i}}(1_{(t_{j-1}^n, t_j^n]})\right) \\ &= \sum_{i=1}^d \left[\delta^{B^{H_i,K_i}}\left(\sum_{j=1}^n \frac{\partial f}{\partial x_i}\left(B_{t_{j-1}^n}^{H,K}\right) 1_{(t_{j-1}^n, t_j^n]}(.)\right)\right. \\ &\left. + \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i^2}\left(B_{t_{j-1}^n}^{H,K}\right)\langle 1_{(0,t_{j-1}^n]}, 1_{(t_{j-1}^n, t_j^n]}\rangle_{\mathcal{H}}\right] \\ &= \sum_{i=1}^d \left[I_1^{n,i} + I_2^{n,i}\right]. \end{aligned}$$

As the similar way in the above theorem, we obtain that for every $i = 1, ..., d$

$$I_2^{n,i} \underset{n\to\infty}{\longrightarrow} H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(B_s^{H,K}\right) s^{2H_iK_i - 1} ds \text{ in } L^2(\Omega).$$

We show that for every $i \in \{1, ..., d\}$

$$I_1^{n,i} \underset{n\to\infty}{\longrightarrow} \int_0^t \frac{\partial f}{\partial x_i}\left(B_s^{H,K}\right)\delta B_s^{H_i,K_i} \text{ in } L^2(\Omega).$$

We set

$$u_s^{n,i} = \sum_{j=1}^n \frac{\partial f}{\partial x_i}\left(B_{t_{j-1}}^{H,K}\right)1_{(t_{j-1},t_j]}(s) - \frac{\partial f}{\partial x_i}\left(B_s^{H,K}\right)1_{(0,t]}(s).$$

By inequality (1.4), we have

$$E\left(\delta^{B^{H_i,K_i}}(u^{n,i})\right)^2 \leq E\|u^{n,i}\|^2_{|\mathcal{H}^i|} + E\|Du^{n,i}\|^2_{|\mathcal{H}^i|\otimes|\mathcal{H}^i|}$$

where $\mathcal{H}^i$ is the Hilbert space associated to B^{H_i,K_i} and R_i its covariance function.
For every $r, s \leq t$

$$D_r u_s^{n,i} = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i^2}\left(B_{t_{j-1}}^{H,K}\right)1_{(0,t_{j-1}]}(r)1_{(t_{j-1},t_j]}(s) - \frac{\partial^2 f}{\partial x_i^2}(B_s^{H,K})1_{(0,s]}(r)$$

we remark that $D_r u_s^{n,i}$ and $u_s^{n,i}$ converge to zero as $n \longrightarrow \infty$ for any $r, s \leq t$. Using (1.12), the Lebesgue dominated convergence theorem and the expression of the norm $\left|\mathcal{H}^i\right| \otimes \left|\mathcal{H}^i\right|$ we obtain that

$$\delta^{B^{H_i,K_i}}(u^{n,i}) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2(\Omega).$$

The proof is thus complete. ■

One can easily generalize the above theorem to the case when the function f depends on time.

Theorem 4. *Let $f \in C^{1,2}\left([0,T]\times\mathbb{R}^d, \mathbb{R}\right)$ and $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ be a d-dimensional bifBm with $2H_iK_i > 1$ for any $i = 1, ..., n$. Assume that the function $f(t,\cdot)$ satisfies (1.12) uniformly in t. Then for every i one has $\frac{\partial f}{\partial x_i}\left(\cdot, B_\cdot^{H,K}\right) \in Dom(\delta^{B^{H_i,K_i}})$ and for every t*

$$\begin{aligned} f\left(t, B_t^{H,K}\right) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s^{H,K})ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}\left(s, B_s^{H,K}\right)\delta B_s^{H_i,K_i} \\ &\quad + \sum_{i=1}^d H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(s, B_s^{H,K}\right)s^{2H_iK_i-1}ds. \end{aligned} \tag{1.28}$$

We consider twice of the kernel of the $d-$dimensional Newtonian potential

$$U(z) = \begin{cases} -\frac{\Gamma(d/2-1)}{2\pi^{d/2}}\frac{1}{|z|^{d-2}} & \text{if } d \geq 3 \\ \frac{1}{\pi}log|z| & \text{if } d = 2. \end{cases}$$

Set

$$\bar{U}(s,z) = \frac{1}{\prod_{j=1}^{d}\sqrt{2H_jK_j}} s^{\theta} U\left(\frac{(z_1-x_1)}{\sqrt{2H_1K_1}}s^{1/2-H_1K_1},...,\frac{(z_d-x_d)}{\sqrt{2H_dK_d}}s^{1/2-H_dK_d}\right) \quad (1.29)$$

where $x=(x_1,...,x_d)\in\mathbb{R}^d$ and $0<\gamma := \frac{1}{2}(2-d)+\theta+(d-2)(HK)^*-\sum_{i=1}^d H_iK_i$ with $(HK)^* = max\{H_1K_1,\dots,H_dK_d\}$.

We shall prove the following Tanaka formula. It will involve a multidimensional weighted local time which is an extension of the one-dimensional local time given by (1.24). Note for any dimension $d\geq 2$ the local time is not a random variable anymore and it is a distribution in the Watanabe's sense.

Theorem 5. *Let $\bar{U}$ as above and let $B^{H,K} = \left(B^{H_1,K_1},...,B^{H_d,K_d}\right)$ be a d-dimensional bifBm with $2H_iK_i>1$ for any $i=1,...d$. Then the following formula holds in the Watanabe space $\mathbb{D}_2^{\alpha-1}$ for any $\alpha<\frac{1}{2(HK)^*}-d/2$.*

$$\bar{U}(t,B_t^{H,K}) = \bar{U}(0,0)+\int_0^t \partial_s\bar{U}(s,B_s^{H,K})ds+\sum_{i=1}^d\int_0^t\frac{\partial\bar{U}(s,B_s^{H,K})}{\partial x_i}\delta B_s^{H_i,K_i}+L^{\theta}(t,x) \quad (1.30)$$

where the generalized weighted local time $L^{\theta}(t,x)$ is defined as

$$L^{\theta}(t,x) = \sum_{n=(n_1,\dots,n_d)}\int_0^t\prod_{i=1}^d\frac{p_{s^{2H_iK_i}}(x_i)}{s^{\frac{1}{2}+(n_i-1)H_iK_i}}H_{n_i}\left(\frac{x_i}{\sqrt{s^{2H_iK_i}}}\right)I_{n_i}^i(1_{[0,s]}^{\otimes n_i})s^{\theta}ds.$$

Proof: We regularize the function $\bar{U}$ by standard convolution. Put $\bar{U}_\varepsilon = p_\varepsilon^d * \bar{U}$, with p_ε^d is the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\varepsilon^d(x) = \prod_{i=1}^d p_\varepsilon(x_i) = \prod_{i=1}^d\frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{x^2}{2\varepsilon}}, \quad \forall x=(x_1,...,x_d)\in\mathbb{R}^d.$$

Using the above Itô formula we have

$$\begin{aligned}\bar{U}_\varepsilon\left(t,B_t^{H,K}\right) &= \bar{U}_\varepsilon(0,0)+\int_0^t\frac{\partial\bar{U}_\varepsilon}{\partial s}\left(s,B_s^{H,K}\right)ds+\sum_{i=1}^d\int_0^t\frac{\partial\bar{U}_\varepsilon}{\partial x_i}\left(s,B_s^{H,K}\right)\delta B_s^{H_i,K_i}\\ &\quad+\sum_{i=1}^d H_iK_i\int_0^t\frac{\partial^2\bar{U}_\varepsilon}{\partial x_i^2}\left(s,B_s^{H,K}\right)s^{2H_iK_i-1}ds.\\ &= \bar{U}_\varepsilon(0,0)+I_1^\varepsilon(t)+I_2^\varepsilon(t).\end{aligned}$$

On the other hand, if $V(z)=U(a_1z_1,...,a_dz_d)$ and $V_\varepsilon = p_\varepsilon^d * V$ we have

$$\frac{1}{2}\sum_{i=1}^d\frac{1}{a_i^2}\frac{\partial^2 V_\varepsilon}{\partial z_i^2}(z) = p_\varepsilon^d(a_1z_1,...,a_dz_d).$$

Hence

$$I_2^{\varepsilon}(t) = \frac{1}{\prod_{j=1}^{d}\sqrt{2H_jK_j}} \int_0^t p_{\varepsilon}^d\left(c_1(s)(B_s^{H_1,K_1}-x_1),...,c_d(s)(B_s^{H_d,K_d}-x_d)\right) s^{\theta} ds$$

where $c_i(s) = \frac{s^{1/2-H_iK_i}}{\sqrt{2H_iK_i}}$ for every $i = 1,...,d$. The next step is to find the chaotic expansion of the last term I_2^{ε}. By Stroock formula, we have

$$p_{\varepsilon}\left(c_i(s)(B_s^{H_i,K_i}-x_i)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n^i\left(ED_{\cdot}^n p_{\varepsilon}\left(c_i(s)(B_s^{H_i,K_i}-x_i)\right)\right)$$

and

$$\begin{aligned}
&ED_{\cdot}^n p_{\varepsilon}\left(c_i(s)(B_s^{H_i,K_i}-x_i)\right) = c_i(s)^n E p_{\varepsilon}^{(n)}\left(c_i(s)(B_s^{H_i,K_i}-x_i)\right) 1_{[0,s]}^{\otimes n}(.) \\
= \quad & c_i(s)^n n!\left(\frac{\varepsilon}{c_i(s)^2}+s^{2H_iK_i}\right)^{-n/2} p_{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}(x_i) \frac{H_n\left(\frac{x_i}{\sqrt{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}}\right)}{c_i(s)^{n+1}} 1_{[0,s]}^{\otimes n}(.) \\
= \quad & \frac{n!}{c_i(s)}\left(\frac{\varepsilon}{c_i(s)^2}+s^{2H_iK_i}\right)^{-n/2} p_{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}(x_i) H_n\left(\frac{x_i}{\sqrt{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}}\right) 1_{[0,s]}^{\otimes n}(.) \\
:= \quad & \frac{n!}{c_i(s)} \beta_{n,\varepsilon}^i(s) 1_{[0,s]}^{\otimes n}(.)
\end{aligned}$$

Consequently

$$p_{\varepsilon}^d\left(c_1(s)(B_s^{H_1,K_1}-x_1),...,c_d(s)(B_s^{H_d,K_d}-x_d)\right) = \sum_{n=(n_1,...,n_d)\in\mathbb{N}^d} \prod_{i=1}^{d} \frac{\beta_{n_i,\varepsilon}^i(s)}{c_i(s)} I_{n_i}^i(1_{[0,s]}^{\otimes n_i})$$

and

$$\begin{aligned}
I_2^{\varepsilon}(t) &= \sum_{n=(n_1,...,n_d)\in\mathbb{N}^d} \int_0^t \prod_{i=1}^{d} \frac{\beta_{n_i,\varepsilon}^i(s)}{s^{\frac{1}{2}-H_iK_i}} I_{n_i}^i(1_{[0,s]}^{\otimes n_i}) s^{\theta} ds \\
&= \sum_{n=(n_1,...,n_d)} \int_0^t \prod_{i=1}^{d} \frac{p_{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}(x_i)}{\left(\frac{\varepsilon}{c_i(s)^2}+s^{2H_iK_i}\right)^{n_i/2} s^{\frac{1}{2}-H_iK_i}} H_{n_i}\left(\frac{x_i}{\sqrt{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}}\right) I_{n_i}^i(1_{[0,s]}^{\otimes n_i}) s^{\theta} ds.
\end{aligned}$$

This term (in fact, slightly modified) appeared in some other papers such as Proposition 12 in [34], or in [113]. Using standard arguments we obtain that the last term converges in $\mathbb{D}_2^{\alpha}$ to $L^{\theta}(t,x)$ as ε goes to 0, with $\alpha < \frac{1}{2(HK)^*} - d/2$.

The rest of the proof consists to show that the following convergences are hold:
For every $i = 1,...,d$

$$\int_0^t \partial_i \bar{U}_{\varepsilon}\left(s, B_s^{H,K}\right) \delta B_s^{H_i,K_i} \overset{\mathbb{D}_2^{\alpha-1}}{\underset{\varepsilon\to 0}{\longrightarrow}} \int_0^t \partial_i \bar{U}\left(s, B_s^{H,K}\right) \delta B_s^{H_i,K_i}. \tag{1.31}$$

$$\int_0^t \partial_s \bar{U}_\varepsilon(s, B_s^{H,K})ds \xrightarrow[\varepsilon\to 0]{\mathbb{D}_2^\alpha} \int_0^t \partial_s \bar{U}(s, B_s^{H,K})ds \tag{1.32}$$

and

$$\bar{U}_\varepsilon(t, B_t^{H,K}) \xrightarrow[\varepsilon\to 0]{\mathbb{D}_2^\alpha} \bar{U}(t, B_t^{H,K}). \tag{1.33}$$

We start with the convergence (1.31). Fix $i \in \{1, ..., d\}$, we note $g_\varepsilon^i(s,z) = \partial_i \bar{U}_\varepsilon(s,z)$. By the formal relation (δ is the Dirac distribution)

$$\int_{\mathbb{R}} f(y)\delta(x-y)dy = f(x)$$

we can write (this is true in the sense of Watanabe distributions)

$$g_\varepsilon^i(s, B_s^{H,K}) = \int_{\mathbb{R}^d} g_\varepsilon^i(s,y)\delta\left(B_s^{H,K} - y\right)dy.$$

Furthermore (see [34] but it can be also derived from a general formula in [63])

$$\begin{aligned}
\delta\left(B_s^{H,K} - y\right) &= \prod_{i=1}^d \delta\left(B_s^{H_i,K_i} - y_i\right) \\
&= \prod_{i=1}^d \left(\sum_{n\geq 0} \frac{1}{(R_i(s))^{n/2}} p_{R_i(s)}(y_i) H_n\left(\frac{y_i}{R_i(s)^{1/2}}\right) I_n^i\left(1_{[0,s]}^{\otimes n}\right)\right) \\
&= \sum_{n=(n_1,...,n_d)} A_n(s,y) I_n(1_{[0,s]}^{\otimes |n|})
\end{aligned}$$

where $R_i(s) = R_i(s,s) = s^{2H_iK_i}$, $A_n(s,y) = \prod_{i=1}^d \frac{1}{(R_i(s))^{n_i/2}} p_{R_i(s)}(y_i) H_{n_i}(\frac{y_i}{R_i(s)^{1/2}})$ and $I_n(1_{[0,s]}^{\otimes|n|}) := \prod_{i=1}^d I_{n_i}^i\left(1_{[0,s]}^{\otimes n_i}\right)$ for every $n = (n_1, ..., n_d)$.
Hence

$$\begin{aligned}
g_\varepsilon^i(s, B_s^{H,K}) &= \sum_{n=(n_1,...,n_d)} \int_{\mathbb{R}^d} g_\varepsilon^i(s,y) A_n(s,y) dy I_n(1_{[0,s]}^{\otimes|n|}) \\
&:= \sum_{n=(n_1,...,n_d)} B_n^{\varepsilon,i}(s) I_n(1_{[0,s]}^{\otimes|n|})
\end{aligned}$$

and using the chaotic form of the divergence integral

$$\begin{aligned}
J_i^\varepsilon(t) &:= \int_0^t g_\varepsilon^i\left(s, B_s^{H,K}\right)\delta B_s^{H_i,K_i} \\
&= \sum_{n=(n_1,...,n_d)} I_{n_i+1}^i\left[B_n^{\varepsilon,i}(s) 1_{[0,s]}^{\otimes n_i}(s_1,...,s_{n_i}) 1_{[0,t]}(s) \prod_{\substack{j=1\\ j\neq i}}^d I_{n_j}^j\left(1_{[0,s]}^{\otimes n_j}\right)\right]^{(s)} \\
&= \sum_{n=(n_1,...,n_d)} I_{n_i+1}^i\left[f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i},s)\right] \\
&= \sum_{n_i\geq 0} I_{n_i+1}^i\left[\sum_{n=(n_1,...,\hat{n}_i,...,n_d)} f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i},s)\right]
\end{aligned}$$

where the superscript (s) denoted the symmetrization with respect to $s_1, \ldots, s_{n_i}, s$, and

$$f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i+1}) = \sum_{l=1}^{n_i+1} \frac{1}{n_i+1} B_n^{\varepsilon,i}(s_l) 1_{[0,s_l]}^{\otimes n_i}(s_1,...,\widehat{s_l},...,s_{n_i+1}) 1_{[0,t]}(s_l) \prod_{\substack{j=1\\ j\neq i}}^{d} I_{n_j}^j\left(1_{[0,s_l]}^{\otimes n_j}\right).$$

Observe here that, since the components of the vector $B^{H,K}$ are independent, the term $\prod_{\substack{j=1\\ j\neq i}}^{d} I_{n_j}^j\left(1_{[0,s_l]}^{\otimes n_j}\right)$ is viewed as a deterministic function for the integral $I_{n_i}^i$. The convergence (1.31) is satisfied if the conditions i) and ii) of Lemma 3 in [34] hold. It is easy to verify the condition i), we will prove only the condition ii).
Fixing $i \in \{1,...,d\}$, we can write,

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\leq \sum_{m\geq 1}(m+1)^{\alpha-1} \sum_{|n|=n_1+...+n_d=m-1} (n_i+1)! E\left\|f_{i,n}^{\varepsilon,t}\right\|_{\mathcal{H}^{\otimes n_i+1}}^2 \\
&= \sum_{m\geq 1}(m+1)^{\alpha-1} \sum_{|n|=n_1+...+n_d=m-1} (n_i+1)! \\
&\times \int_{[0,T]^{n_i+1}}\int_{[0,T]^{n_i+1}} \sum_{l,k=1}^{n_i+1} \frac{1}{(n_i+1)^2} |B_n^{\varepsilon,i}(s_l)||B_n^{\varepsilon,i}(r_k)| 1_{[0,t]}(s_l) 1_{[0,t]}(r_k) \\
&\times 1_{[0,s_l]}^{\otimes n_i}(s_1,...,\widehat{s_l},...,s_{n_i+1}) 1_{[0,r_k]}^{\otimes n_i}(r_1,...,\widehat{r_l},...,r_{n_i+1}) \prod_{\substack{j\neq i\\ j=1}}^{d} n_j! R_j(s_l,r_k)^{n_j} \\
&\times \prod_{q=1}^{n_i+1} \frac{\partial^2 R_i}{ds_q dr_q}(s_q,r_q) dr_1....dr_{n_i+1} ds_1....ds_{n_i+1}
\end{aligned}$$

Since $\alpha < 0$ then $(m+2)^{\alpha-1} \leq (m+1)^{\alpha-1}$ and $(n_i+1) \leq (m+1)$. This implies that

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\leq \sum_{m\geq 0}(m+1)^{\alpha} \sum_{|n|=n_1+...+n_d=m} (n_i)! \Bigg[(1 - \frac{1}{(n_i+1)}) \\
&\times \int_{[0,T]^2}\int_{[0,T]^2} |B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_2)| 1_{[0,t]}(s_1) 1_{[0,t]}(r_2) R_i(s_1,r_2)^{n_i-1} \\
&\times \prod_{\substack{j\neq i\\ j=1}}^{d} n_j! R_j(s_1,r_2)^{n_j} 1_{[0,s_1]}(s_2) 1_{[0,r_2]}(r_1) \frac{\partial^2 R_i}{ds_1 dr_1}(s_1,r_1) \frac{\partial^2 R_i}{ds_2 dr_2}(s_2,r_2) ds_1 ds_2 dr_1 dr_2 \\
&+ \frac{1}{(n_i+1)} \int_{[0,T]^2}\int_{[0,T]^2} |B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_1)| 1_{[0,t]}(s_1) 1_{[0,t]}(r_1) R_i(s_1,r_1)^{n_i-1} \\
&\times \prod_{\substack{j\neq i\\ j=1}}^{d} n_j! R_j(s_1,r_1)^{n_j} 1_{[0,s_1]}(s_2) 1_{[0,r_1]}(r_2) \frac{\partial^2 R_i}{ds_1 dr_1}(s_1,r_1) \frac{\partial^2 R_i}{ds_2 dr_2}(s_2,r_2) ds_1 ds_2 dr_1 dr_2 \Bigg].
\end{aligned}$$

By integration we obtain

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\le \sum_{m\ge 0}(m+1)^\alpha \sum_{|n|=n_1+\ldots+n_d=m}(n_i)!\Bigg[(1-\frac{1}{(n_i+1)})\\
&\times \int_{[0,t]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_2)|R_i(s_1,r_2)^{n_i-1}\\
&\times \prod_{\substack{j\neq i\\ j=1}}^d n_j!R_j(s_1,r_2)^{n_j}\frac{\partial R_i}{ds_1}(s_1,r_2)\frac{\partial R_i}{dr_2}(s_1,r_2)ds_1dr_2\\
&+ \frac{1}{(n_i+1)}\int_{[0,t]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_1)|R_i(s_1,r_1)^{n_i}\\
&\times \prod_{\substack{j\neq i\\ j=1}}^d n_j!R_j(s_1,r_1)^{n_j}\frac{\partial^2 R_i}{ds_1dr_1}(s_1,r_1)ds_1dr_1\Bigg].
\end{aligned}$$

We have for any $1/4\le\beta\le 1/2$

$$\begin{aligned}
B_n^{\varepsilon,i}(s) &= \int_{\mathbb{R}^d} g_\varepsilon^i(s,y)A_n(s,y)dy\\
&= \int_{\mathbb{R}^d} g_\varepsilon^i(s,y)\prod_{j=1}^d H_{n_j}\left(\frac{y_j}{\sqrt{R_j(s)}}\right)e^{\frac{-\beta y_j^2}{R_j(s)}}\frac{1}{\sqrt{R_j(s)}^{n_j}}\frac{e^{-(\frac12-\beta)\frac{y_j^2}{R_j(s)}}}{\sqrt{2\pi R_j(s)}}dy\\
&= \int_{\mathbb{R}^d} p_\varepsilon^d(z)dz\int_{\mathbb{R}^d}\partial_i\bar U\left(s,y-z\right)\prod_{j=1}^d H_{n_j}\left(\frac{y_j}{\sqrt{R_j(s)}}\right)\frac{e^{\frac{-\beta y_j^2}{R_j(s)}}}{\sqrt{2\pi}}\frac{e^{-(\frac12-\beta)\frac{y_j^2}{R_j(s)}}}{\sqrt{R_j(s)}^{n_j+1}}dy.
\end{aligned}$$

Since (see Lemma 11 in [34])

$$\sup_{z\in\mathbb{R}^d}\prod_{j=1}^d|H_{n_j}\left(z_j\right)|e^{-\beta z_j^2}\le C\prod_{j=1}^d\frac{1}{\sqrt{n_j!}(n_j\vee 1)^{\frac{8\beta-1}{12}}}$$

and for every $(s,z)\in(0,T]\times\mathbb{R}^d$

$$\left|\partial_i\bar U(s,z)\right|\le Cs^{\frac12(1-d)+\theta}\left|\left((z_1-x_1)s^{-H_1K1},...,(z_d-x_d)s^{-H_dKd}\right)\right|^{1-d}.$$

Then, for any $s\in(0,T]$

$$\begin{aligned}
|B_n^{\varepsilon,i}(s)| &\le C\prod_{j=1}^d\frac{1}{\sqrt{n_j!}(n_j\vee 1)^{\frac{8\beta-1}{12}}}\frac{1}{s^{n_jH_jK_j}}\int_{\mathbb{R}^d}p_\varepsilon^d(z)dz\\
&\times \int_{\mathbb{R}^d}s^{\frac12(1-d)+\theta-\sum_{j=1}^d H_jK_j}\frac{e^{-(\frac12-\beta)|s^{-HK}y|^2}}{|s^{-HK}(y-z-x)|^{d-1}}dy\\
&\le Cs^{\gamma-\frac12}\prod_{j=1}^d\frac{1}{\sqrt{n_j!}(n_j\vee 1)^{\frac{8\beta-1}{12}}}\frac{1}{s^{n_jH_jK_j}}\int_{\mathbb{R}^d}p_\varepsilon^d(z)dz\\
&\times \int_{\mathbb{R}^d}\frac{e^{-(\frac12-\beta)|s^{-HK}y|^2}}{|(y-z-x)|^{d-1}}dy
\end{aligned}$$

where $s^{-HK}y := (s^{-H_1K_1}y_1, ..., s^{-H_dK_d}y_d)$§.
Let η a positive constant such that, for every $s \in (0,T]$, $j \in \{1,\dots,d\}$ we have $s^{-2H_jK_j} > \eta$.
Combining this with for any $a, b \in \mathbb{R}$, $a^2 \geq \frac{1}{2}(a-b)^2 - b^2$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} p_\varepsilon^d(z)dz \int_{\mathbb{R}^d} \frac{e^{-(\frac{1}{2}-\beta)|s^{-HK}y|^2}}{|(y-z-x)|^{d-1}}dy &\leq \int_{\mathbb{R}^d} p_\varepsilon^d(z)dz \int_{\mathbb{R}^d} \frac{e^{-\eta(\frac{1}{2}-\beta)|y|^2}}{|(y-z-x)|^{d-1}}dy \\
&\leq \int_{\mathbb{R}^d} p_\varepsilon^d(z) e^{\eta(\frac{1}{2}-\beta)|z+x|^2}dz \int_{\mathbb{R}^d} \frac{e^{-\frac{\eta}{2}(\frac{1}{2}-\beta)|y-(z+x)|^2}}{|(y-(z+x))|^{d-1}}dy \\
&\leq Ce^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} p_\varepsilon^d(z) e^{2\eta(\frac{1}{2}-\beta)|z|^2}dz \\
&\leq Ce^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} \frac{e^{-\frac{|v|^2}{2}}}{\sqrt{2\pi}^d} e^{2\eta(\frac{1}{2}-\beta)\varepsilon|v|^2}dv \\
&\leq Ce^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} \frac{e^{-\beta|v|^2}}{\sqrt{2\pi}^d}dv < \infty
\end{aligned}$$

since $2\eta\varepsilon \leq 1$ when ε is close to 0.
Thus

$$|B_n^{\varepsilon,i}(s)| \leq Cs^{\gamma-\frac{1}{2}} \prod_{j=1}^d \frac{1}{\sqrt{n_j!}(n_j \vee 1)^{\frac{8\beta-1}{12}}} \frac{1}{s^{n_jH_jK_j}}$$

where C is a constant depending only on $d,\ H,\ K,\ T,\ x$ and β.
On the other hand, there exist a constant $C(H,K)$ positive, such that for every $i = 1,\dots,d$

$$\left|\frac{\partial R_i}{dr}(r,s)\frac{\partial R_i}{ds}(r,s)\right| \leq C(H,K)(rs)^{2H_iK_i-1},$$

$$\left|\frac{\partial^2 R_i}{drds}(r,s)\right| \leq C(H,K)(rs)^{H_iK_i-1}$$

and

$$\left|\frac{R_i(r,s)}{(rs)^{H_iK_i}}\right| \leq C(H,K).$$

It follows by anterior inequalities that

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\leq C\sum_{m\geq 0}(1+m)^\alpha \sum_{|n|=n_1+\ldots+n_d=m} \prod_{j=1}^d \frac{1}{(n_j \vee 1)^{\frac{8\beta-1}{6}}} \\
&\times \int_{[0,t]^2} \frac{R_i(r,s)^{n_i-1}}{(rs)^{(n_i-1)H_iK_i}}(rs)^{\gamma-\frac{3}{2}+H_iK_i} \prod_{\substack{j=1\\ j\neq i}}^d \frac{R_j(r,s)^{n_j}}{(rs)^{n_jH_jK_j}}drds.
\end{aligned}$$

We use the selfsimilarity of the covariance kernel $R(r,s) = R(1,\frac{s}{r})r^{2HK}$ and the change of variables $r/s = z$ in the integral respect to dz to obtain

$$\begin{aligned}\|J_i^{\varepsilon}(t)\|^2_{\alpha-1,2} &\leq C\sum_{m\geq 0}(1+m)^{\alpha}\sum_{|n|=n_1+\ldots+n_d=m}\prod_{j=1}^{d}\frac{1}{(n_j\vee 1)^{\frac{8\beta-1}{6}}}\\ &\times\int_0^t r^{2(\gamma-1+H_iK_i)}dr\int_0^1(z)^{\gamma-\frac{3}{2}+H_iK_i}\left(\frac{R_i(1,z)}{z^{H_iK_i}}\right)^{n_i-1}\prod_{\substack{j=1\\ j\neq i}}^{d}\left(\frac{R_j(1,z)}{z^{H_jK_j}}\right)^{n_j}dz.\end{aligned}$$

Since for each $i\in\{1,\ldots,d\}$, $\gamma-\frac{3}{2}+H_iK_i>-1$, $2(\gamma-1+H_iK_i)>-1$ and from Lemma 12 and the proof of the Proposition 12 in [34], we obtain that

$$\begin{aligned}\|J_i^{\varepsilon}(t)\|^2_{\alpha-1,2} &\leq C\sum_{m\geq 0}(1+m)^{\alpha}m^{-\frac{1}{2(HK)^*}}\sum_{|n|=n_1+\ldots+n_d=m}\prod_{j=1}^{d}\frac{1}{(n_j\vee 1)^{\frac{8\beta-1}{6}}}\\ &\leq C\sum_{m\geq 0}(1+m)^{\alpha}(m)^{-\frac{1}{2(HK)^*}-1+d(1-\frac{8\beta-1}{6})}\end{aligned}$$

is finite if and only if $\alpha<\frac{1}{2(HK)^*}-\frac{d}{2}$.
On the other hand, for every $(s,z)\in(0,T]\times\mathbb{R}^d$ we have

$$\left|\partial_s\bar{U}(s,z)\right|\leq Cs^{\frac{-d}{2}+\theta}\left|\left((z_1-x_1)s^{-H_1K1},...,(z_d-x_d)s^{-H_dKd}\right)\right|^{2-d}.$$

and

$$\left|\bar{U}(s,z)\right|\leq Cs^{\frac{1}{2}(2-d)+\theta}\left|\left((z_1-x_1)s^{-H_1K1},...,(z_d-x_d)s^{-H_dKd}\right)\right|^{2-d}$$

and this inequalities imply as in [115] the convergences (1.32) and (1.33) in $\mathbb{D}^{\alpha}_2$. ■

Chapter 2

Occupation densities for certain processes related to fractional Brownian motion

In this paper we establish the existence of a square integrable occupation density for two classes of stochastic processes. First we consider a Gaussian process with an absolutely continuous random drift, and secondly we handle the case of a (Skorohod) integral with respect to the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The proof of these results uses a general criterion for the existence of a square integrable local time, which is based on the techniques of Malliavin calculus.

2.1 Introduction

Local times for semimartingales have been widely studied. See for example the monograph [101] and the references therein. On the other hand, local times of Gaussian processes have also been the object of a rich probabilistic literature; see for example the recent book [74] by Marcus and Rosen. A general criterion for the existence of a local time for a wide class of anticipating processes, which are not semimartingales nor Gaussian processes, was established by Imkeller and Nualart in [50]. The proof of this result combines the techniques of Malliavin calculus with the criterion given by Geman and Horowitz in [44]. This criterion was applied in [50] to the Brownian motion with an anticipating drift, and to indefinite Skorohod integral processes.

The aim of this paper is to establish the existence of the occupation density for two classes of stochastic processes related to the fractional Brownian motion, using the approach introduced in [50]. First we consider a Gaussian process $B = \{B_t, t \in [0,1]\}$ with an absolutely continuous random drift

$$X_t = B_t + \int_0^t u_s ds,$$

where u is a stochastic process measurable with respect to the σ-field generated by B. We assume that the variance of the increment of the Gaussian process B on an interval $[s,t]$ behaves as

$|t-s|^{2\rho}$, for some $\rho \in (0,1)$. This includes, for instance, the bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$ and the fractional Brownian motion in the particular case $K = 1$. Under reasonable regularity hypotheses imposed to the process u we prove the existence of a square integrable occupation density with respect to the Lebesgue measure for the process X.

Our second example is the indefinite divergence (Skorohod) integral $X = \{X_t, t \in [0,1]\}$ with respect to the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, that is

$$X_t = \int_0^t u_s \delta B_s^H.$$

We provide integrability conditions on the integrand u and its iterated derivatives in the sense of Malliavin calculus in order to deduce the existence of a square integrable occupation density for X.

We organize our paper as follows. Section 2 contains some preliminaries on the Malliavin calculus with respect to Gaussian processes. In Section 3 we prove the existence of the occupation densities for perturbed Gaussian processes and in Section 4 we treat the case of indefinite divergence integral processes with respect to the fractional Brownian motion.

2.2 Preliminaries

Let $\{B_t, t \in [0,1]\}$ be a centered Gaussian process with covariance function

$$R(t,s) := E(B_t B_s),$$

defined in a complete probability space $(\Omega, \mathcal{F}, P)$. We assume that the σ-field $\mathcal{F}$ is generated by B. By $\mathcal{H}$ we denote the canonical Hilbert space associated to B defined as the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0,1]\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R(t,s), \qquad s,t \in [0,1].$$

The mapping $\mathbf{1}_{[0,t]} \to X_t$ can be extended to an isometry between $\mathcal{H}$ and the first Gaussian chaos generated by B. We denote by $B(\varphi)$ the image of an element $\varphi \in \mathcal{H}$ by this isometry.

We will first introduce some elements of the Malliavin calculus associated with B. We refer to [81] for a detailed account of these notions. For a smooth random variable $F = f(B(\varphi_1), \dots, B(\varphi_n))$, with $\varphi_i \in \mathcal{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded) the derivative of F with respect to B is defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), \dots, B(\varphi_n))\varphi_j.$$

For any integer $k \geq 1$ and any real number $p \geq 1$ we denote by $\mathbb{D}^{k,p}$ the Sobolev space defined as the closure of the space of smooth random variables with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^k \|D^j F\|_{L^p(\Omega;\mathcal{H}^{\otimes j})}^p,$$

where D^j denotes the iterated derivative operator of D. Similarly, for a given Hilbert space V we can define Sobolev spaces of V-valued random variables $\mathbb{D}^{k,p}(V)$.

The following lemma (see Lemma 1.2.3 in [81]) is a useful tool in proving that a given random variable belongs to the Sobolev space $\mathbb{D}^{1,p}$, where $p > 1$.

Lemma 3. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables converging in L^p, $p > 1$, to F. Suppose that* $\sup_n E\left(\|DF_n\|_{\mathcal{H}}^p\right) < \infty$. *Then, F belongs to $\mathbb{D}^{1,p}$.*

Consider the adjoint δ of D in L^2. Its domain is the class of elements $u \in L^2(\Omega; \mathcal{H})$ such that

$$E(\langle DF, u\rangle_{\mathcal{H}}) \leq C\|F\|_{1,2},$$

for any $F \in \mathbb{D}^{1,2}$, and $\delta(u)$ is the unique element of $L^2(\Omega)$ given by

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathcal{H}})$$

for any $F \in \mathbb{D}^{1,2}$. We will make use of the notation $\delta(u) = \int_0^1 u_s \delta B_s$, although, in general, the trajectories of u are not real-valued functions. It is well-known that $\mathbb{D}^{1,2}(\mathcal{H})$ is included in the domain of δ. Note that $E(\delta(u)) = 0$ and the variance of $\delta(u)$ is given by

$$E(\delta(u)^2) = E(\|u\|_{\mathcal{H}}^2) + E(\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}), \tag{2.1}$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H})$, where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H}\otimes\mathcal{H}$. Meyer's inequality tells us that

$$E(|\delta(u)^p|) \leq C_p\left(E(\|u\|_{\mathcal{H}}^p) + E(\|Du\|_{\mathcal{H}\otimes\mathcal{H}}^p)\right), \tag{2.2}$$

for any $p > 1$. We will make use of the property

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}. \tag{2.3}$$

if $F \in \mathbb{D}^{1,2}$, $u \in \text{Dom}(\delta)$ and the random elements Fu and $F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}$ are square integrable. We also need the commutativity relationship between D and δ

$$D\delta(u) = u + \int_0^1 Du_s \delta B_s, \tag{2.4}$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the $\mathcal{H}$-valued process $\{Du_s, s \in [0,1]\}$ belongs to the domain of δ.

Throughout this paper we will assume that the centered Gaussian process $B = \{B_t, t \in [0,1]\}$ satisfies

$$C_1|t-s|^{2\rho} \leq E(|B_t - B_s|^2) \leq C_2|t-s|^{2\rho}, \tag{2.5}$$

for some $\rho \in (0,1)$ with C_1, C_2 two positive constants not depending on t, s. It will follow from the Kolmogorov criterium that B admits a Hölder continuous version of order δ for any $\delta < \rho$.

Throughout this paper we will denote by C a generic constant that may be different from line to line.

Example 1. *The bifractional Brownian motion (see, for instance [47]), denoted by $B^{H,K}$, is defined as a centered Gaussian process with covariance*

$$R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \tag{2.6}$$

where $H \in (0,1)$ and $K \in (0,1]$. When $K = 1$, then we have a standard fractional Brownian motion denoted by B^H. It has been proven in [47] that for all $s \le t$,

$$2^{-K}|t-s|^{2HK} \le E\left(\left|B_t^{H,K} - B_s^{H,K}\right|^2\right) \le 2^{1-K}|t-s|^{2HK}. \tag{2.7}$$

So, relation (2.5) holds with $\rho = HK$. A stochastic analysis for this process can be found in [61] and a study of its occupation density has been done in [39], [113].

For a measurable function $x : [0,1] \to \mathbb{R}$ we define the occupation measure

$$\mu(x)(C) = \int_0^1 \mathbf{1}_C(x_s)ds,$$

where C is a Borel subset of $\mathbb{R}$ and we will say that x has an occupation density with respect to the Lebesgue measure λ if the measure μ is absolutely continuous with respect to λ. The occupation density of the function x will be the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$. For a continuous process $\{X_t, t \in [0,1]\}$ we will say that X has an occupation density on $[0,1]$ if for almost all $\omega \in \Omega$, $X(\omega)$ has an occupation density on $[0,1]$.

We will use the following criterion for the existence of occupation densities (see [50]). Set $T = \{(s,t) \in [0,1]^2 : s < t\}$.

Theorem 6. *Let $\{X_t, t \in [0,1]\}$ be a continuous stochastic process such that $X_t \in \mathbb{D}^{2,2}$ for every $t \in [0,1]$. Suppose that there exists a sequence of bounded random variables $\{F_n, n \ge 1\}$ with $\bigcup_n\{F_n \ne 0\} = \Omega$ a.s. and $F_n \in \mathbb{D}^{1,1}$ for every $n \ge 1$, two sequences $\alpha_n > 0, \delta_n > 0$, a measurable bounded function $\gamma : [0,1] \to \mathbb{R}$, such that $\gamma D(X_t - X_s) \in \mathcal{H}$, $\gamma^{\otimes 2}D^2(X_t - X_s) \in \mathcal{H}\otimes\mathcal{H}$ for all $0 \le s < t \le 1$, $\gamma DF_n \in \mathcal{H}$ for all $n \ge 1$ and a constant $\theta > 0$, such that:*

a) *For every $n \ge 1$, $|t-s| \le \delta_n$, and on $\{F_n \ne 0\}$ we have*

$$\langle \gamma D(X_t - X_s), \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}} > \alpha_n|t-s|^\theta, \qquad \text{a.s..} \tag{2.8}$$

b) *For every $n \ge 1$*

$$\int_T E(\langle \gamma DF_n, \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}})|t-s|^{-\theta}dtds < \infty. \tag{2.9}$$

c) *For every $n \ge 1$*

$$\int_T E\left(\left|F_n\left\langle \gamma^{\otimes 2}DD(X_t - X_s), \mathbf{1}_{(s,t]}^{\otimes 2}\right\rangle_{\mathcal{H}^{\otimes 2}}\right|\right)|t-s|^{-2\theta}dsdt < \infty. \tag{2.10}$$

Then the process $\{X_t, t \in [0,1]\}$ admits a square integrable occupation density on $[0,1]$.

Remark 2. *The original result has been stated in [50] with $\theta = 1$ in the case of the standard Brownian motion. On the other hand, by applying Proposition 2.3 and Theorem 2.1 in [50] it follows easily that this criterion can be stated for any $\theta > 0$.*

2.3 Occupation density for Gaussian processes with random drift

We study in this part the existence of the occupation density for Gaussian processes perturbed by an absolute continuous random drift. The main result of this section is the following.

Theorem 7. *Let $\{B_t, t \in [0,1]\}$ be a centered Gaussian process satisfying (2.5). Consider the process $\{X_t, t \in [0,1]\}$ given by*

$$X_t = B_t + \int_0^t u_s ds,$$

and suppose that the process u satisfies the following conditions:

i) $u \in \mathbb{D}^{2,2}(L^2([0,1]))$.

ii) $E\left(\left(\int_0^1 \|D^2 u_t\|^p_{\mathcal{H}\otimes\mathcal{H}}\, dt\right)^{\frac{\lambda}{p}}\right) < \infty$, *for some $\lambda > 1$ and $p > \frac{1}{1-\rho}$.*

Then, the process X has a square integrable occupation density on the interval $[0,1]$.

Proof: We are going to apply Theorem 6. Notice first that $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0,1]$ due to Condition i). Let us first prove Condition a) in Theorem 6. We assume that $\gamma = 1$. For any $0 \le s < t \le 1$, applying the derivative operator to the increment $X_t - X_s$ yields

$$D(X_t - X_s) = \mathbf{1}_{(s,t]} + \int_s^t Du_r dr.$$

As a consequence, using (2.5) we can write

$$\begin{aligned} \left\langle D(X_t - X_s), \mathbf{1}_{(s,t]}\right\rangle_{\mathcal{H}} &= \left\langle \mathbf{1}_{(s,t]}, \mathbf{1}_{(s,t]}\right\rangle_{\mathcal{H}} + \left\langle \int_s^t Du_r dr, \mathbf{1}_{(s,t]}\right\rangle_{\mathcal{H}} \\ &= E(|B_t - B_s|^2) + \left\langle \int_s^t Du_r dr, \mathbf{1}_{(s,t]}\right\rangle_{\mathcal{H}} \\ &\geq C_1(t-s)^{2\rho} - \left|\left\langle \int_s^t Du_r dr, \mathbf{1}_{[s,t]}\right\rangle_{\mathcal{H}}\right|. \end{aligned} \tag{2.1}$$

Applying again (2.5) and Hölder's inequality yields

$$\begin{aligned} \left|\left\langle \int_s^t Du_r dr, \mathbf{1}_{[s,t]}\right\rangle_{\mathcal{H}}\right| &\leq \left\| \int_s^t Du_r dr\right\|_{\mathcal{H}} \|\mathbf{1}_{[s,t]}\|_{\mathcal{H}} \\ &\leq \sqrt{C_2}(t-s)^{1-\frac{1}{p}+\rho} \left(\int_0^1 \|Du_r\|^p_{\mathcal{H}}\, dr\right)^{\frac{1}{p}}. \end{aligned} \tag{2.2}$$

Fix a natural number $n \geq 2$, and choose a function $\varphi_n(x)$, which is infinitely differentiable with compact support, such that $\varphi_n(x) = 1$ if $|x| \leq n-1$, and $\varphi_n(x) = 0$, if $|x| \geq n$. Set $F_n = \varphi_n(G)$, where

$$G = \left(\int_0^1 \|Du_r\|^p_{\mathcal{H}}\, dr\right)^{\frac{1}{p}}.$$

The sequence of random variables F_n satisfies $\bigcup_n \{F_n \neq 0\} = \Omega$, because for each $n \geq 2$, the set $\{F_n \neq 0\}$ is included in $\{G \leq n\}$. We claim that for each n, the random variable F_n belongs to $\mathbb{D}^{1,\lambda}$. In fact, by duality we can write

$$G = \sup_{h \in B_q} \int_0^1 \langle Du_r, h_r \rangle_{\mathcal{H}} \, dr,$$

where B_q is the unit ball in the Banach space $h \in L^q([0,1]; \mathcal{H})$, and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{\mathcal{D}_N, N \geq 1\}$ be an increasing sequence of finite subsets of B_q, whose union is dense in B_q. Set

$$G_N = \sup_{h \in \mathcal{D}_N} \int_0^1 \langle Du_r, h_r \rangle_{\mathcal{H}} \, dr.$$

Clearly, $\varphi_n(G_N)$ converges to F_n, as N tends to infinity, in L^λ. On the other hand, for each N, the random variable G_N belongs to $\mathbb{D}^{1,2}$, because it is a supremum of a finite number of random variables in $\mathbb{D}^{1,2}$, and the supremum is a Lipschitz function. Finally, from Lemma 3, it suffices to show that the norms $\|D(\varphi_n(G_N))\|_{\mathcal{H}}$ are uniformly bounded in $L^\lambda(\Omega)$, which follows from Condition ii):

$$\begin{aligned}
\|D(\varphi_n(G_N))\|_{\mathcal{H}} &= \|\varphi_n'(G_N) DG_N\|_{\mathcal{H}} \leq \|\varphi_n'\|_\infty \sup_{h \in \mathcal{D}_N} \left\| \int_0^1 \langle D^2 u_r, h_r \rangle_{\mathcal{H}} \, dr \right\|_{\mathcal{H}} \\
&\leq \|\varphi_n'\|_\infty \sup_{h \in \mathcal{D}_N} \sup_{\substack{v \in \mathcal{H} \\ \|v\|_{\mathcal{H}} \leq 1}} \left| \int_0^1 \langle D^2 u_r, v \otimes h_r \rangle_{\mathcal{H}^{\otimes 2}} \, dr \right| \\
&\leq \|\varphi_n'\|_\infty \left(\int_0^1 \|D^2 u_r\|_{\mathcal{H}^{\otimes 2}}^p \, dr \right)^{\frac{1}{p}} \in L^\lambda(\Omega).
\end{aligned}$$

Choose the constants $\alpha_n = \frac{C_1}{2}$, and $\delta_n = \left(\frac{C_1}{2n\sqrt{C_2}} \right)^{\frac{p}{p-1-p\rho}}$. Then, from (2.1) and (2.2), on the set $\{F_n \neq 0\} \subset \{G \leq n\}$, assuming $|t-s| \leq \delta_n$, and taking into account that $1 - \frac{1}{p} - \rho > 0$, we get

$$\begin{aligned}
\left\langle D(X_t - X_s), \mathbf{1}_{(s,t]} \right\rangle_{\mathcal{H}} &\geq C_1 (t-s)^{2\rho} - n\sqrt{C_2} (t-s)^{1-\frac{1}{p}+\rho} \\
&= (t-s)^{2\rho} \left[C_1 - n\sqrt{C_2} (t-s)^{1-\frac{1}{p}-\rho} \right] \\
&\geq \alpha_n (t-s)^{2\rho},
\end{aligned}$$

and property a) of Theorem 6 holds with $\theta = 2\rho$.

Condition b) follows immediately from (2.5):

$$\int_T \frac{E\left(\left| \left\langle DF_n, \mathbf{1}_{(s,t]} \right\rangle_{\mathcal{H}} \right| \right)}{|t-s|^{2\rho}} dsdt \leq \sqrt{C_2} \int_T \frac{E\left(\|DF_n\|_{\mathcal{H}} \right)}{|t-s|^{\rho}} dsdt < \infty,$$

Finally, Condition c) can also be checked:

$$\begin{aligned}
&\int_T \frac{E\left(\left|F_n \left\langle D^2(X_t - X_s), \mathbf{1}^{\otimes 2}_{([s,t]}\right\rangle_{\mathcal{H}^{\otimes 2}}\right|\right)}{|t-s|^{4\rho}} dsdt \\
\leq\ & \|\varphi_n\|_\infty \int_T \frac{E\left(\left\|D^2(X_t - X_s)\right\|_{\mathcal{H}^{\otimes 2}}\right) \left\|1^{\otimes 2}_{(s,t]}\right\|_{\mathcal{H}^{\otimes 2}}}{|t-s|^{4\rho}} dsdt \\
=\ & \|\varphi_n\|_\infty \int_T \frac{E\left(\left\|D^2(X_t - X_s)\right\|_{\mathcal{H}^{\otimes 2}}\right) E(B_t - B_s)^2}{|t-s|^{4\rho}} dsdt \\
\leq\ & C_2 \|\varphi_n\|_\infty \int_T \frac{E\left(\left\|D^2(X_t - X_s)\right\|_{\mathcal{H}^{\otimes 2}}\right)}{|t-s|^{2\rho}} dsdt < \infty,
\end{aligned}$$

because

$$\begin{aligned}
E\left(\left\|D^2(X_t - X_s)\right\|_{\mathcal{H}^{\otimes 2}}\right) &= E\left(\left\|\int_s^t D^2 u_r dr\right\|_{\mathcal{H}^{\otimes 2}}\right) \leq E\left(\int_s^t \left\|D^2 u_r\right\|_{\mathcal{H}^{\otimes 2}} dr\right) \\
&\leq (t-s)^{1-\frac{1}{p}} E\left[\left(\int_0^1 \left\|D^2 u_r\right\|^p_{\mathcal{H}^{\otimes 2}} dr\right)^{\frac{1}{p}}\right],
\end{aligned}$$

and $1 - \frac{1}{p} - 2\rho > -1$, taking into account that $p > \frac{1}{2(1-\rho)}$. ■

Remark 3. *These conditions are intrinsic and they do not depend on the structure of the Hilbert space $\mathcal{H}$. In the case of the Brownian motion, this result is slightly weaker than Theorem 3.1 in [50], because we require here a little more integrability.*

2.4 Occupation density for Skorohod integrals with respect to the fractional Brownian motion

We study here the existence of occupation densities for indefinite divergence integrals with respect to the fractional Brownian motion. Consider a process of the form $X_t = \int_0^t u_s \delta B_s^H$, $t \in [0,1]$, where B is fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, and u is an element of $\mathbb{D}^{1,2}(\mathcal{H}) \subset \mathrm{Dom}\,(\delta)$.

We know that the covariance of the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ can be written as

$$E(B_t^H B_s^H) = \int_0^t \int_0^s \phi(\alpha,\beta) d\alpha d\beta, \tag{2.1}$$

where $\phi(\alpha,\beta) = H(2H-1)|\alpha-\beta|^{2H-2}$. For any $0 \leq s < t \leq 1$, and $\alpha \in [0,1]$ we set

$$f_{s,t}(\alpha) := \int_s^t \phi(\alpha,\beta) d\beta. \tag{2.2}$$

We also know (see e.g. [81]) that the canonical Hilbert space associated to B^H satisfies:

$$L^2([0,1]) \subset L^{\frac{1}{H}}([0,1]) \subset \mathcal{H}, \tag{2.3}$$

and the scalar product in $\mathcal{H}$ is given by

$$\langle f,g\rangle_{\mathcal{H}} = \int_0^1\int_0^1 f(s)g(t)\phi(s,t)dsdt, \tag{2.4}$$

if $f,g \in L^{\frac{1}{H}}([0,1])$.

The following is the main result of this section.

Theorem 8. *Consider the stochastic process $X_t = \int_0^t u_s\delta B_s^H$, where the integrand u satisfies the following conditions for some $q>2$ and $p > \frac{2q}{H(q-2)}$:*

1) $u \in \mathbb{D}^{3,2}(L^2([0,1]))$.

2) $\int_0^1\int_0^1 [E(|D_t u_s|^p) + E(\| \,|D_t D u_s|\, \|_{\mathcal{H}}^p) + E(\| \,|D_t D^2 u_s|\, \|_{\mathcal{H}\otimes\mathcal{H}}^p)]dsdt < \infty$.

3) $\int_0^1 E\left(|u_t|^{-\frac{p}{p-1}(q+1)}\right)dt < \infty$.

Then the process $\{X_t, t\in[0,1]\}$ admits a square integrable occupation density on $[0,1]$.

Proof: We will apply again Theorem 6. Condition 1) implies that $X_t \in \mathbb{D}^{2,2}$ for all $t\in[0,1]$. On the other hand, from Theorem 7.8 in [61] (or also by a slightly modification of Theorem 5 in [2]) we obtain the continuity of the paths of the process X. Note that from Lemma 2.2 in [50] corroborated with Hypothesis 3) we obtain the existence of a function $\gamma:[0,1]\to\{-1,1\}$ such that $\gamma_t u_t = |u_t|$ for almost all t and ω. Note that the fact that the function γ is bounded and the hypothesis $u\in\mathbb{D}^{3,2}(L^2([0,1]))$ and $H>\frac{1}{2}$ imply $\gamma D(X_t - X_s)\in\mathcal{H}$, $\gamma^{\otimes 2}D^2(X_t-X_s)\in\mathcal{H}\otimes\mathcal{H}$ for all $0\le s<t\le 1$. We are going to show conditions a), b) and c) of Theorem 6.

Proof of condition a): Fix $0\le s<t\le 1$. From (2.4) we obtain

$$D(X_t - X_s) = u\mathbf{1}_{(s,t]} + \int_s^t Du_r\delta B_r^H,$$

and we can write

$$\langle \gamma D(X_t-X_s),\ \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}} = \langle |u|\mathbf{1}_{(s,t]}, \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}} + \langle \int_s^t Du_r\delta B_r^H, \gamma\mathbf{1}_{(s,t]}\rangle_{\mathcal{H}}. \tag{2.5}$$

We need a lower bound for the first summand in the above expression and an upper bound for the absolute value of the second summand. Using (2.4) the first summand can be written as

$$\langle |u|\mathbf{1}_{(s,t]}, \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}} = \int_s^t\int_s^t |u_\alpha|\phi(\alpha,\beta)d\alpha d\beta = \int_s^t |u_\alpha| f_{s,t}(\alpha)d\alpha.$$

From the identity

$$\begin{aligned} E(|B_t^H - B_s^H|^2) &= \int_s^t f_{s,t}(\alpha)d\alpha \\ &= \int_s^t (|u_\alpha|f_{s,t}(\alpha))^{\frac{q}{q+1}}(|u_\alpha|f_{s,t}(\alpha))^{-\frac{q}{q+1}} f_{s,t}(\alpha)d\alpha, \end{aligned}$$

and applying Hölder's inequality with orders $\frac{q+1}{q}$ and $q+1$, we obtain

$$(t-s)^{2H} \leq \left(\int_s^t |u_\alpha| f_{s,t}(\alpha) d\alpha \right)^{\frac{q}{q+1}} \left(\int_s^t |u_\alpha|^{-q} f_{s,t}(\alpha) d\alpha \right)^{\frac{1}{q+1}}. \tag{2.6}$$

Notice that the function $f_{s,t}(\alpha)$ is bounded

$$f_{s,t}(\alpha) \leq f_{0,1}(\alpha) = H(2H-1) \int_0^1 |\alpha - \beta|^{2H-2} d\beta = H\left(\alpha^{2H-1} + (1-\alpha)^{2H-1}\right) \leq 2H.$$

Hence, from (2.6) we get

$$\int_s^t |u_\alpha| f_{s,t}(\alpha) d\alpha \geq C|t-s|^{\frac{2H(q+1)}{q}} Z_q^{-\frac{1}{q}}, \tag{2.7}$$

where $Z_q = \int_0^1 |u_\alpha|^{-q} d\alpha$.

On the other hand, for the second summand in the right-hand side of (2.5) we can write, using (2.4) and Hölder's inequality.

$$\begin{aligned} \left| \left\langle \gamma \int_s^t D u_r \delta B_r^H, \mathbf{1}_{(s,t]} \right\rangle_{\mathcal{H}} \right| &= \left| \int_s^t \int_0^1 \left(\int_s^t D_\alpha u_r \delta B_r^H \right) \gamma(\beta) \phi(\alpha, \beta) d\alpha d\beta \right| \\ &\leq \int_0^1 \left| \int_s^t D_\alpha u_r \delta B_r^H \right| f_{s,t}(\alpha) d\alpha \\ &\leq \left(\int_0^1 f_{s,t}(\alpha)^{\frac{p}{p-1}} d\alpha \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_0^1 \left| \int_s^t D_\alpha u_r \delta B_r^H \right|^p d\alpha \right)^{\frac{1}{p}}. \end{aligned} \tag{2.8}$$

We claim that

$$\left(\int_0^1 f_{s,t}(\alpha)^{\frac{p}{p-1}} d\alpha \right)^{\frac{p-1}{p}} \leq C|t-s|^\eta, \tag{2.9}$$

for some $\eta < 2H - \frac{1}{p}$. In fact. the right-hand side of (2.9) can be written as

$$\begin{aligned} \left(\int_0^1 f_{s,t}(\alpha)^{\frac{p}{p-1}} d\alpha \right)^{\frac{p-1}{p}} &= c_H \left\| \int_s^t |\cdot - \beta|^{2H-2} d\beta \right\|_{L^{\frac{p}{p-1}}([0,1])} \\ &\leq c_H \left\| \mathbf{1}_{[s,t]} * |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})}, \end{aligned} \tag{2.10}$$

where $c_H = H(2H-1)$. Young's inequality with exponents a and b in $(1, \infty)$ such that $\frac{1}{a} + \frac{1}{b} = 2 - \frac{1}{p}$ yields

$$\left\| \mathbf{1}_{[s,t]} * |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})} \leq \left\| \mathbf{1}_{[s,t]} \right\|_{L^a(\mathbb{R})} \left\| |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^b(\mathbb{R})}. \tag{2.11}$$

Choosing $b < \frac{1}{2-2H}$ and letting $\eta = \frac{1}{a} < 2H - \frac{1}{p}$ we obtain (2.9).

On the other hand, applying Lemma II.2.4 in [62] (see Nikol'skiĭ [78], and Garsia-Rodemich-Rumsey [43]) with exponents p and $m>0$, and to the continuous function $u_s = \int_0^s D_\alpha u_r \delta B_r^H$, yields

$$\left|\int_s^t D_\alpha u_r \delta B_r^H\right|^p \leq N|t-s|^m \int_0^1\int_0^1 \frac{\left|\int_x^y D_\alpha u_r \delta B_r^H\right|^p}{|x-y|^{m+2}} dxdy,$$

for some constant $N>0$. As a consequence

$$\left(\int_0^1 \left|\int_s^t D_\alpha u_r \delta B_r^H\right|^p d\alpha\right)^{\frac{1}{p}} \leq N^{\frac{1}{p}}|t-s|^{\frac{m}{p}} Y_{m,p}^{\frac{1}{p}}, \tag{2.12}$$

where

$$Y_{m,p} = \int_0^1\int_0^1\int_0^1 \frac{\left|\int_x^y D_\alpha u_r \delta B_r^H\right|^p}{|x-y|^{m+2}} dxdyd\alpha.$$

Substituting (2.9) and (2.12) into (2.8) yields

$$\left|\left\langle \gamma \int_s^t D u_r \delta B_r^H, \mathbf{1}_{(s,t]}\right\rangle_{\mathcal{H}}\right| \leq C|t-s|^{\eta+\frac{m}{p}} Y_{m,p}^{\frac{1}{p}}, \tag{2.13}$$

and from (2.7), (2.13) and (2.5) we get

$$\begin{aligned}\langle \gamma D(X_t - X_s), \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}} &\geq C\left(|t-s|^{\frac{2H(q+1)}{q}} Z_q^{-\frac{1}{q}} - |t-s|^{\eta+\frac{m}{p}} Y_{m,p}^{\frac{1}{p}}\right) \\ &= C|t-s|^{2H+\frac{2H}{q}}\left(Z_q^{-\frac{1}{q}} - |t-s|^{\delta} Y_{m,p}^{\frac{1}{p}}\right),\end{aligned} \tag{2.14}$$

where $\delta = \eta + \frac{m}{p} - 2H - \frac{2H}{q}$. If we assume that

$$m > 1 + \frac{2Hp}{q}, \tag{2.15}$$

then we can choose $\eta < 2H - \frac{1}{p}$ in such a way that the exponent δ is positive.

We construct now the sequence $\{F_n, n\geq 1\}$. Fix a natural number $n\geq 2$, and choose a function $\varphi_n(x)$, which is infinitely differentiable with compact support, such that $\varphi_n(x)=1$ if $|x|\leq n-1$, and $\varphi_n(x)=0$, if $|x|\geq n$. Set $F_n = \varphi_n(G)$, where $G = Z_q + Y_{m,p}$. Then, from (2.14) and choosing the constants $\alpha_n = \frac{1}{2n^{1/q}}$ and $\delta_n = \frac{1}{2n^{\frac{1}{p}+\frac{1}{q}}}$, it follows that Condition a) of Theorem 6 holds with $\theta = 2H + \frac{2H}{q}$.

It only remains to show that the random variables F_n are in the space $\mathbb{D}^{1,1}$. This will give that $\gamma DF_n \in \mathcal{H}$ for every n. We can approximate Z_q and $Y_{m,p}$ by the sequences

$$Z_q^N = \int_0^1 \left(u_s^2 + \frac{1}{N}\right)^{-\frac{q}{2}} ds,$$

and

$$Y_{m,p}^N = \int_0^1\int_0^1\int_0^1 \frac{\left|\int_x^y D_\alpha u_r \delta B_r^H\right|^p}{|x-y|^{m+2} + \frac{1}{N}} dxdyd\alpha.$$

Clearly, $\varphi_n(Z_q^N + Y_{m,p}^N)$ converges to F_n, as N tends to infinity, almost surely and in L^1. Thus, to show that F_n belongs to $\mathbb{D}^{1,1}$ it suffices to prove that the derivatives $D\left(\varphi_n(Z_q^N + Y_{m,p}^N)\right)$

converge in $L^1(\Omega;\mathcal{H})$ as N tends to infinity. The random variables Z_q^N and $Y_{m,p}^N$ belong to $\mathbb{D}^{1,1}$ and we can write

$$DZ_q^N = -q\int_0^1 \left(u_s^2+\frac{1}{N}\right)^{-\frac{q}{2}-1} u_s Du_s ds,$$

and

$$DY_{m,p}^N = p\int_0^1\int_0^1\int_0^1 \frac{|\xi_{x,y,\alpha}|^{p-1}\operatorname{sign}(\xi_{x,y,\alpha})D\xi_{x,y,\alpha}}{|x-y|^{m+2}+\frac{1}{N}}dxdyd\alpha,$$

where $\xi_{x,y,\alpha}=\int_y^x D_\alpha u_r\delta B_r$. As a consequence, $D\left(\varphi_n(Z_q^N+Y_{m,p}^N)\right)$ converges pointwise as N tends to infinity to

$$\varphi_n'(Z_q+Y_{m,p})\left[DZ_q+DY_{m,p}\right],$$

where

$$DZ_q=(-q)\int_0^1 |u_s|^{-q-1}\operatorname{sign}(u_s)Du_s ds,$$

and

$$DY_{m,p}=p\int_0^1\int_0^1\int_0^1 |\xi_{x,y,\alpha}|^{p-1}\operatorname{sign}(\xi_{x,y,\alpha})D\xi_{x,y,\alpha}|x-y|^{-m-2}dxdyd\alpha,$$

Then, suffices to show that the random variables $\|DZ_q\|_{\mathcal{H}}$ and $\|DY_{m,p}\|_{\mathcal{H}}$ are integrable on the set $\{G\le n\}$. By Hölder's inequality

$$E\left(\|DZ_q\|_{\mathcal{H}}\right)\le q\left(\int_0^1 E\left(|u_s|^{-\frac{p}{p-1}(q+1)}\right)ds\right)^{1-\frac{1}{p}}\left(\int_0^1 E\left(\|Du_s\|_{\mathcal{H}}^p\right)ds\right)^{\frac{1}{p}}.$$

The first factor in the right-hand side is finite by Condition 3), and the second factor is also finite by Condition 2), the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into $\mathcal{H}$ (see (2.3), and the fact that $pH\ge 1$. On the other hand,

$$\begin{aligned}\|DY_{m,p}\|_{\mathcal{H}} &\le p\int_0^1\int_0^1\int_0^1 |\xi_{x,y,\alpha}|^{p-1}\|D\xi_{x,y,\alpha}\|_{\mathcal{H}}|x-y|^{-m-2}dxdyd\alpha\\ &\le p(Y_{m,p})^{\frac{p-1}{p}}\left(\int_0^1\int_0^1\int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}}^p|x-y|^{-m-2}dxdyd\alpha\right)^{1/p}.\end{aligned}$$

Then, on the set $\{G\le n\}$, the factor $(Y_{m,p})^{\frac{p-1}{p}}$ is bounded and it suffices to show that the random variable

$$Y=\int_0^1\int_0^1\int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}}^p|x-y|^{-m-2}dxdyd\alpha$$

has a finite expectation. Since, for any $0\le y<x\le 1$

$$D\xi_{x,y,\alpha}=\mathbf{1}_{[y,x]}D_\alpha u+\int_y^x DD_\alpha u_s\delta B_s^H,$$

we have

$$\begin{aligned}Y &\le C\Bigg(\int_0^1\int_0^1\int_0^1 \|\mathbf{1}_{[y,x]}D_\alpha u\|_{\mathcal{H}}^p|x-y|^{-m-2}dxdyd\alpha\\ &\quad+\int_0^1\int_0^1\int_0^1 \|\int_y^x DD_\alpha u_s\delta B_s^H\|_{\mathcal{H}}^p|x-y|^{-m-2}dxdyd\alpha\Bigg)\\ &:= C(Y_1+Y_2).\end{aligned}$$

¿From the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into $\mathcal{H}$, we obtain

$$\begin{aligned} Y_1 &\leq C\int_0^1\int_0^1\int_0^1 \|\mathbf{1}_{[y,x]}D_\alpha u\|^p_{L^{1/H}([0,1])}|x-y|^{-m-2}dxdyd\alpha \\ &\leq C|x-y|^{pH-1}\int_0^1\int_0^1\int_0^1\int_y^x |D_\alpha u_r|^p\,|x-y|^{-m-2}drdxdyd\alpha. \end{aligned}$$

Hence, $E(Y_1)<\infty$, by Fubini's theorem, Proposition 3.1 in [50] and Condition 2), provided

$$m < pH-1. \tag{2.16}$$

On the other hand, using the estimate (2.2), and again the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into $\mathcal{H}$ yields

$$\begin{aligned} E\left(\|\int_y^x DD_\alpha u_s \delta B_s^H\|^p_{\mathcal{H}}\right) &\leq C\ E\left(\left\|D_\alpha Du.\mathbf{1}_{[y,x]}(\cdot)\right\|^p_{\mathcal{H}^{\otimes 2}} + \left\|D_\alpha D^2 u.\mathbf{1}_{[y,x]}(\cdot)\right\|^p_{\mathcal{H}^{\otimes 3}}\right) \\ &\leq C\ E\Big(\left\||D_\alpha Du.|\ \mathbf{1}_{[y,x]}(\cdot)\right\|^p_{L^{1/H}([0,1];\mathcal{H})} \\ &\quad + \left\||D_\alpha D^2 u.|\ \mathbf{1}_{[y,x]}(\cdot)\right\|^p_{L^{1/H}([0,1];\mathcal{H}^{\otimes 2})}\Big) \\ &\leq C|x-y|^{pH-1}\left(\int_y^x E\left(\||D_\alpha Du_r|\|^p_{\mathcal{H}}\right)dr \right. \\ &\quad \left. + \int_y^x E\left(\||D_\alpha D^2 u_r|\|^p_{\mathcal{H}^{\otimes 2}}\right)dr\right). \end{aligned}$$

As before we obtain $E(Y_2)<\infty$ by Fubini's theorem and Condition 2), provided (2.16) holds. Notice that condition $p>\frac{2q}{H(q-2)}$ implies that we can choose an m such that (2.15) and (2.16) hold.

Proof of Condition b): Define $A_n=\{G\leq n\}$. Then, Condition b) in Theorem 6 follows from

$$\begin{aligned} \int_T E(\langle\gamma DF_n, \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}})|t-s|^{-\theta}dtds &\leq C\int_T E(\mathbf{1}_{A_n}\left|\langle\gamma DG, \mathbf{1}_{(s,t]}\rangle_{\mathcal{H}}\right|)|t-s|^{-\theta}dtds \\ &\leq CE\left(\mathbf{1}_{A_n}\|DG\|_{\mathcal{H}}\right)\int_T|t-s|^{H-\theta}dsdt<\infty, \end{aligned}$$

since $E\left(\mathbf{1}_{A_n}\|DG\|_{\mathcal{H}}\right)<\infty$ and $\theta-H=H+\frac{2H}{q}<1$.

Proof of Condition c): We have

$$D_\alpha D_\beta(X_t-X_s)=\mathbf{1}_{(s,t]}(\beta)D_\alpha u_\beta+\mathbf{1}_{(s,t]}(\alpha)D_\beta u_\alpha+\int_s^t D_\alpha D_\beta u_r\delta B_r^H.$$

Hence,

$$\begin{aligned} \left\langle\gamma^{\otimes 2}D^2(X_t-X_s), \mathbf{1}^{\otimes 2}_{(s,t]}\right\rangle_{\mathcal{H}^{\otimes 2}} &= \left\langle\gamma^{\otimes 2}\mathbf{1}_{(s,t]}(\beta)D_\alpha u_\beta, \mathbf{1}^{\otimes 2}_{(s,t]}\right\rangle_{\mathcal{H}^{\otimes 2}} + \left\langle\gamma^{\otimes 2}\mathbf{1}_{(s,t]}(\alpha)D_\beta u_\alpha, \mathbf{1}^{\otimes 2}_{(s,t]}\right\rangle_{\mathcal{H}^{\otimes 2}} \\ &\quad + \left\langle\gamma^{\otimes 2}\int_s^t D_\alpha D_\beta u_r\delta B_r^H, \mathbf{1}^{\otimes 2}_{(s,t]}\right\rangle_{\mathcal{H}^{\otimes 2}} \\ &:= J_1(s,t)+J_2(s,t)+J_3(s,t). \end{aligned}$$

For $i = 1, 2, 3$, we set

$$A_i = E\left(F_n \int_T |t-s|^{-2\theta} |J_i(s,t)|\, dsdt\right).$$

Let us compute first

$$\begin{aligned} A_1 &\leq C \int_T \int_T |t-s|^{2H-2\theta} E\left(\| |D_\alpha u_\beta| \mathbf{1}_{(s,t]}(\beta) \|_{\mathcal{H}^{\otimes 2}}\right) dsdt \\ &= C \int_T \int_T |t-s|^{2H-2\theta} \left(\int_s^t \int_s^t \varphi(\beta, y) d\beta dy\right)^{\frac{1}{2}} dsdt, \end{aligned}$$

where

$$\varphi(\beta, y) = \int_0^1 \int_0^1 E\left(|D_\alpha u_\beta| |D_x u_y|\right) \phi(\alpha, x) \phi(\beta, y) d\alpha dx.$$

By Fubini's theorem $A_1 < \infty$, because $2H - 2\theta > -2$, which is equivalent to $q > H$, and

$$\int_0^1 \int_0^1 \varphi(\beta, y) d\beta dy \leq E\left(\| |Du| \|^2_{\mathcal{H}^{\otimes 2}}\right)$$

and this is finite because of the inclusion of $L^2([0,1])$ in $\mathcal{H}$. In the same way we can show that $A_2 < \infty$. Finally,

$$\begin{aligned} A_3 &= E\left(F_n \int_T \int_T |t-s|^{-2\theta} \left| \left\langle \gamma^{\otimes 2} \int_s^t D_\alpha D_\beta u_r \delta B_r^H, \mathbf{1}_{(s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} \right| dsdt\right) \\ &\leq C \int_T \int_T |t-s|^{2H-2\theta} E\left(\left\| \int_s^t DDu_r \delta B_r^H \right\|_{\mathcal{H}^{\otimes 2}}\right) dsdt, \end{aligned}$$

and we conclude as before by using, for example, the bound (2.2) for the norm of the Skorohod integral and Condition 2). ■

Remark 4. *If the process u is bounded below, then we can choose any q, and the condition on p is just $p > \frac{2}{H}$.*

Part II

LIMIT THEOREMS FOR ROSENBLATT PROCESS AND STEIN ESTIMATION ON GAUSSIAN SPACE

Chapter 3

Non-central limit theorem for the cubic variation of a class of selfsimilar stochastic processes

By using multiple Wiener-Itô stochastic integrals, we study the cubic variation of a class of selfsimilar stochastic processes with stationary increments (the Rosenblatt process with selfsimilarity order $H \in \left(\frac{1}{2}, 1\right)$). This study is motivated by statistical purposes. We prove that this renormalized cubic variation satisfies a non-central limit theorem and its limit is (in the $L^2(\Omega)$ sense) still the Rosenblatt process.

3.1 Introduction

The self-similarity property for a stochastic process means that scaling of time is equivalent to an appropriate scaling of space. That is, a process $(Y_t)_{t\geq 0}$ is selfsimilar of order $H > 0$ if for all $c > 0$ the processes $(Y_{ct})_{t\geq 0}$ and $(c^H Y_t)_{t\geq 0}$ have the same finite dimensional distributions. The selfsimilar processes are of interest for various applications, such as economics, internet traffic of hydrology. The fractional Brownian motion (fBm) is the usual candidate to model phenomena in which the selfsimilarity property can be observed from the empirical data. Recall that the fractional Brownian motion is a centered Gaussian process with covariance function $R^H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. The parameter $H \in (0,1)$ characterizes almost all the important properties of the process. The fBm can be also defined as the only centered Gaussian process which is selfsimilar with stationary increments. In some models the gaussianity assumption could be not plausible and in this case one needs to use a different selfsimilar process with stationary increments to model the phenomena. Natural candidates are the Hermite processes: these stochastic processes appear as limits in the so-called Non-Central Limit Theorem (see [20], [32], [108], [45]). In contrast with the classical Central Limit Theorem, the non-central limit theorem deals with sequences of dependent random variable whose renomalized sum converges in some situations to a non gaussian distribution. For a complete exposition of limit theorems in probability theory, we refer to [56] or [98]. Except the Gaussian character, these Hermite processes have the same property as the fBm with Hurst parameter $H > \frac{1}{2}$: selfsimilarity, stationarity of increments, Hölder continuous path, long -range dependence. While the fractional Brownian motion can be expressed as a Wiener integral with respect to the standard Wiener process, the Hermite process of order $q \geq 2$ is a q iterated integral of a deterministic function with q variables with respect to the Brownian motion. The Rosenblatt process is obtained in the particular case $q = 2$. It will be properly defined in Section 2. This processes have been recently studied by several authors (see [18], [22], [80], [75], [79], [111], [112]).

The Hurst parameter H characterizes all the important properties of a Hermite process, as seen above. Therefore, estimating H properly is of the utmost importance. Several statistics have been introduced to this end, such as wavelets, k-variations, variograms, maximum likelihood estimators, or spectral methods. Information on these various approaches can be found in the book of Beran [6].

One of the most popular methods to estimate the selfsimilarity order for stochastic process is based on the study of their variations. The p-variation of a process $(X_t)_{t\in[0,1]}$ is defined as the limit of the sequence (sometimes the absolute value of the increment is used in the definition)

$$V^{p,N}(X) = \frac{1}{N}\left[\sum_{i=0}^{N-1} \frac{\left(X_{\frac{i+1}{N}} - X_{\frac{i}{N}}\right)^p}{\mathbf{E}\left(X_{\frac{i+1}{N}} - X_{\frac{i}{N}}\right)^p} - 1\right]. \tag{3.1}$$

There exists a direct connection between the behavior of the variations and the convergence of an estimator for the selfsimilarity order based of these variation (see [23], [112]); basically if there renormalized variation satisfies a central limit theorem then the estimator satisfies a central limit theorem and this fact is very useful for statistical aspects.

In a recent paper ([112]) the quadratic variation of the Rosenblatt process $(Z_t^{(H)})_{t\in[0,1]}$ with selfsimilarity order $H \in (\frac{1}{2}, 1)$ has been studied. The following facts happen: the normalized sequence $N^{1-H}V^{2,N}(Z^{(H)})$ satisfies a non-central limit theorem, it converges in L^2 to the Rosenblatt random variable $Z_1^{(H)}$. From this, we can construct an estimator for H whose behavior is still non-normal. This situation is somehow not good for statistical applications because one always prefers the estimators which are asymptotically normal. To have normal estimators we need to define some adjusted variations (as in [112]).

In the fractional Brownian motion case the well-known non-normality of the quadratic variation when $H \in (\frac{3}{4}, 1)$ can be avoided by using "longer filters" (that means, replacing the increments $X_{\frac{i+1}{N}} - X_{\frac{i}{N}}$ by $X_{\frac{i+1}{N}} - 2X_{\frac{i}{N}} + X_{\frac{i-1}{N}}$) or higher order variations (choosing a bigger p). In this work we will consider the second choice (the first choice will be treated in a separate paper): we replace the quadratic variation by the cubic variation for the Rosenblatt processes to see what happens and if it is possible to find a Gaussian distribution as law of the renormalized cubic variation. In the fractional Brownian motion case, this has no sense because the third moment of a centered Gaussian random variable is zero. In this paper we therefore study the cubic variation for this process. We use the Wiener chaos expansion for the statistics $V^{3,N}(Z^{(H)})$

and we will decompose it in several terms in the Wiener chaoses of ordres $2, 4$ and 6. As in other cases ([112], [22]) the second chaos term is dominant and it has to be renormalized by N^{1-H} to have a non-trivial limit. We note that the rate of convergence N^{1-H} is the same as for quadratic variation, so there no gain for the speed and moreover the limit is again, modulo a constant, a Rosenblatt random variable with index H (only the constant is changing). This property has been called in [22] the *reproduction property* of the Rosenblatt process because its variations generates again Rosenblatt random variable as limits. We conjecture that the same property holds true for the p-power variations.

The organization of our paper is as follows. Section 2 contains the presentation of the basic tools that we will need throughout the paper: multiple Wiener-Itô integrals and their basic properties, the definition of the Rosenblatt process and its characteristics. In Section 3 we estimate the mean square of the cubic variation of the Rosenblatt process and we give its normalization and finally in Section 4 we prove a non-central limit theorem for the renormalized cubic variation.

3.2 Preliminaries

3.2.1 Multiple stochastic integrals

In this paragraph we describe the basic elements of calculus on Wiener chaos. Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. We refer to [81] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.

Let $f \in \mathcal{S}_m$ be an elementary functions with m variables that can be written as

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_1}\times\ldots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoints. For a such step function f we define

$$I_m(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1})\ldots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that the application I_n constructed above from $\mathcal{S}$ to $L^2(\Omega)$ is is linear and the following properties hold:

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{3.2}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

It also holds that

$$I_n(f) = I_n\left(\tilde{f}\right)$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\sigma\in S_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Since the set $\mathcal{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geq 1$ the mapping I_n can be extended to a linear and continuous operator from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 dW_{t_n} \int_0^{t_n} \dots \int_0^{t_2} f(t_1,\dots,t_n) dW_{t_1};$$

here the integrals are of Itô type; this formula is easy to show for elementary f's, and follows for general $f \in L^2([0,1]^n)$ by a density argument.

The product for two multiple integrals can be expanded into a sum of multiple integrals (see [81]): if $f \in L^2([0,1]^n)$ and $g \in L^2([0,1]^m)$ are symmetric functions, then it holds that

$$I_n(f)I_m(g) = \sum_{l=0}^{m\wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g) \tag{3.3}$$

where the contraction $f \otimes_l g$ belongs to $L^2([0,1]^{m+n-2l})$ for $l = 0,1,\dots,m\wedge n$ and it is given by

$$\begin{aligned} &(f \otimes_l g)(s_1,\dots,s_{n-l},t_1,\dots,t_{m-l}) \\ = &\int_{[0,1]^l} f(s_1,\dots,s_{n-l},u_1,\dots,u_l) g(t_1,\dots,t_{m-l},u_1,\dots,u_l) du_1 \dots du_l. \end{aligned} \tag{3.4}$$

When $l = 0$, we will denote, throughout this paper, by $f \otimes g := f \otimes_0 g$.

3.2.2 The Rosenblatt process

The Rosenblatt process $(Z^{(H)}(t))_{t\in[0,1]}$ appears as a limit in the so-called *Non Central Limit Theorem* (see [32], [108], [45]). It is not a Gaussian process and can be defined through its representation as double iterated integral with respect to a standard Wiener process (see [111]). More precisely, the Rosenblatt process with self-similarity order $H \in (\frac{1}{2},1)$ is defined by

$$Z_t^{(H)} := \int_0^t \int_0^t L_t(y_1,y_2) dW_{y_1} dW_{y_2} \tag{3.5}$$

where $(W_t, t \in [0,1])$ is a Brownian motion,

$$L_t^H(y_1,y_2) := L_t(y_1,y_2) = d(H) 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) du, \tag{3.6}$$

with

$$H' := \frac{H+1}{2} \quad \text{and} \quad d(H) := \frac{1}{H+1}\left(\frac{H}{2(2H-1)}\right)^{-\frac{1}{2}}$$

and with K^H the standard kernel defined in (3.7) appearing in the Wiener integral representation of the fBm (for $t > s$ and $H > \frac{1}{2}$)

$$K^H(t,s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \tag{3.7}$$

with $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$ and $\beta(\cdot,\cdot)$ the beta function. The derivative of K^H is

$$\frac{\partial K^H}{\partial t}(t,s) := \partial_1 K^H(t,s) = c_H \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{3.8}$$

The two parameters function L_t given by (3.6) will be called the kernel of the Rosenblatt process. The following key relation in crucial in our calculation and it will repeatedly used in the paper

$$\int_0^{u\wedge v} \partial_1 K^{H'}(u,y)\partial_1 K^{H'}(v,y)dy = a(H)|u-v|^{2H'-2} \tag{3.9}$$

with $a(H) = H'(2H'-1)$. Among the main properties of the Rosenblatt process, we recall

- it is H-self-similar in the sense that for any $c > 0$, $(Z^{(H)}_{ct}) =^{(d)} (c^H Z^{(H)}_t)$, where " $=^{(d)}$ " means equivalence of all finite dimensional distributions;
- it has stationary increments, that is, the joint distribution of $(Z^{(H)}_{t+h} - Z^{(H)}_h, t \in [0,1])$ is independent of $h > 0$.
- $\mathbf{E}(|Z^{(H)}_t|^p) < \infty$ for any $p > 0$, and $Z^{(H)}$ has the same covariance than a standard fractional Brownian motion with parameter H.
- the Rosenblatt process is Hölder continuous, of order $\delta < H$. This can easily obtained by the Kolmogorov continuity criterium.

3.3 Renormalization of the cubic variation

3.3.1 Estimation of the mean square

We will study in this paragraph the cubic variation of the Rosenblatt process obtained by putting $p = 3$ in (3.1)

$$V^{3,N} = \frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right)^3}{\mathbf{E}\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right)^3} - 1\right) \tag{3.10}$$

By denoting for $= 1,\dots,N$

$$f_{i,N} = L^{(H)}_{\frac{i+1}{N}} - L^{(H)}_{\frac{i}{N}}$$

we obtain $Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}} = I_2(f_{i,N})$ where I_2 is a multiple integral of order 2 as defined in Section 2.1 and then

$$V^{3,N} = \frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{(I_2(f_{i,N}))^3}{\mathbf{E}\left(I_2(f_{i,N})\right)^3} - 1\right).$$

By using the product formula for multiple Wiener-Itô integrals (3.3), for any function $f \in L^2([0,1]^2)$ symmetric,

$$\begin{aligned} & I_2(f)^3 \\ = \; & I_6\left((f\tilde{\otimes}f)\otimes f\right) + 8I_4\left((f\tilde{\otimes}f)\otimes_1 f\right) + 4I_4\left((f\otimes_1 f)\otimes f\right) \\ & +12I_2\left((f\tilde{\otimes}f)\otimes_2 f\right) + 16I_2\left((f\otimes_1 f)\otimes_1 f\right) + 2\langle f,f\rangle_{L^2([0,1]^2)}I_2(f) + 8\langle (f\otimes_1 f), f\rangle_{L^2([0,1]^2)}. \end{aligned}$$

Here and in the sequel $f\tilde{\otimes}f$ denotes the symmetrization of the function $f\otimes f$ which is not necessary symmetric even if f is symmetric. Applying this to $f = f_{i,N}$ we obtain

$$(I_2(f_{i,N}))^3 = 8(f_{i,N}\otimes_1 f_{i,N})\otimes_2 f_{i,N} + I_2(g_{i,N}) + 4I_4(h_{i,N}) + I_6((f_{i,N}\tilde{\otimes}f_{i,N})\otimes f_{i,N}) \quad (3.11)$$

Here we used the following notation

$$g_{i,N} = 2\|f_{i,N}\|^2_{L^2} f_{i,N} + 12(f_{i,N}\tilde{\otimes}f_{i,N})\otimes_2 f_{i,N} + 16(f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N} \quad (3.12)$$

and

$$h_{i,N} = 2(f_{i,N}\tilde{\otimes}f_{i,N})\otimes_1 f_{i,N} + f_{i,N}\otimes(f_{i,N}\otimes_1 f_{i,N}) := h^{(1)}_{i,N} + h^{(2)}_{i,N}. \quad (3.13)$$

Note that $g_{i,N} \in L^2([0,1]^2)$ and $h_{i,N} \in L^2([0,1]^4)$. On the other hand, we can simplify a little bit the above expressions since

$$(f_{i,N}\tilde{\otimes}f_{i,N})\otimes_2 f_{i,N} = \frac{1}{3}\|f_{i,N}\|^2_{L^2} f_{i,N} + \frac{2}{3}(f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N}.$$

Hence the kernel of the second chaos term can be written as

$$g_{i,N} = 6\|f_{i,N}\|^2_{L^2} f_{i,N} + 24(f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N}.$$

We start with the following lemma where we compute the cubic mean of the increment of the Rosenblatt process. We already observe a significant difference from the Gaussian case: this cubic mean is not zero.

Lemma 4. *Let $(Z_t^{(H)})_{t\in[0,1]}$ be a Rosenblatt process with selfsimilarity index $H \in (\frac{1}{2},1)$. Then, for every $s,t \in [0,1]$*

$$\mathbf{E}\left(Z_t^{(H)} - Z_s^{(H)}\right)^3 = C(H)|t-s|^{3H} \times sign(t-s) \quad (3.14)$$

where

$$C(H) = 8a(H)^3 d(H)^3 \int_{[0,1]^3} \left(|u-v||u-u'||v-u'|\right)^{2H'-2} du du' dv. \quad (3.15)$$

Proof: Let us denote by

$$f_{s,t}(x,y) = L_t(x,y) - L_s(x,y)$$

where L is the kernel of the Rosenblatt process given by (3.6) and $x, y \in [0,1]$. We will have, by using relation (3.9),

$$\begin{aligned}(f_{s,t} \otimes_1 f_{s,t})(x,y) &= \int_0^1 f_{s,t}(x,z) f_{s,t}(y,z) dz \\ &= d(H)^2 a(H) \Big(1_{[0,t]}(x,y) \int_x^t \int_y^t \partial_1 K^{H'}(u,x) \partial_1 K^{H'}(v,y) |u-v|^{2H'-2} dv du \\ &\quad -1_{[0,t]}(x) 1_{[0,s]}(y) \int_x^t \int_y^s \partial_1 K^{H'}(u,x) \partial_1 K^{H'}(v,y) |u-v|^{2H'-2} dv du \\ &\quad -1_{[0,s]}(x) 1_{[0,t]}(y) \int_x^s \int_y^t \partial_1 K^{H'}(u,x) \partial_1 K^{H'}(v,y) |u-v|^{2H'-2} dv du \\ &\quad + 1_{[0,s]^2}(x,y) \int_x^s \int_y^s \partial_1 K^{H'}(u,x) \partial_1 K^{H'}(v,y) |u-v|^{2H'-2} dv du \Big).\end{aligned}$$

The computation of the cubic mean of a multiple integral in the second chaos (3.11) implies

$$\mathbf{E}\left(Z_t^{(H)} - Z_s^{(H)} \right)^3 = 8 \langle f_{s,t} \otimes_1 f_{s,t}, f_{s,t} \rangle_{L^2([0,1]^2)}.$$

We compute, by (3.9)

$$\begin{aligned}\langle f_{s,t} \otimes_1 f_{s,t}, f_{s,t} \rangle_{L^2([0,1]^2)} &= \int_{[0,1]^2} (f_{s,t} \otimes_1 f_{s,t})(x,y) f_{s,t}(x,y) dx dy \\ &= d(H)^3 a(H)^3 \int_s^t \int_s^t \int_s^t \left(|u-v||u-u'||v-u'| \right)^{2H'-2} du du' dv.\end{aligned}$$

By the change of variables $\bar{u} = \frac{u-s}{t-s}$ we will transform the integrals on $[s,t]$ into integrals from 0 to 1. We immediately obtain the relation (3.14).

To calculate $\mathbf{E}(V^{3,N})^2$ we apply the above result and we obtain

$$\begin{aligned}\mathbf{E}[(I_2(f_{i,N}))^3] &= 8 (f_{i,N} \otimes_1 f_{i,N}) \otimes_2 f_{i,N} \\ &= 8 d(H)^3 a(H)^3 \int_{I_i} \int_{I_i} \int_{I_i} dy_1 dy_2 dy_3 \left(|y_1 - y_2| \cdot |y_2 - y_3| \cdot |y_3 - y_1| \right)^{2H'-2} \\ &= 8 \frac{d(H)^3 a(H)^3}{N^{6H'-3}} \int_{[0,1]^3} dy_1 dy_2 dy_3 \left(|y_1 - y_2| \cdot |y_2 - y_3| \cdot |y_3 - y_1| \right)^{2H'-2} \\ &= C(H) N^{-(6H'-3)} = C(H) N^{-3H}.\end{aligned}$$

where $a(H) = \frac{H(H+1)}{2}$ and $C(H)$ is defined in (3.15).

We can write the expression of the statistics V_N as follows

$$\begin{aligned}V^{3,N} &= \frac{1}{C(H) N^{1-3H}} \sum_{i=0}^{N-1} \left((I_2(f_{i,N}))^3 - \mathbf{E}\left(I_2(f_{i,N}) \right)^3 \right) \\ &= \frac{1}{C(H) N^{1-3H}} \sum_{i=0}^{N-1} \left(I_2(g_{i,N}) + 4 I_4\left(h_{i,N} \right) + I_6((f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N}) \right). \qquad (3.16)\end{aligned}$$

We prove next the following renormalization result.

Proposition 5. *Let $V^{3,N}$ the cubic variation statistics of the Rosenblatt process. Then*

$$\mathbf{E}\left(N^{1-H}V^{3,N}\right)^2 \to_{N\to\infty} \bar{C}(H) \tag{3.17}$$

where $\bar{C}(H) := C(H)^2 C_0(H)$ *with*

$$C_0(H) = \Big(9 + 36C'(H)H(2H-1) + 144\left[C'(H)H(2H-1)\right]^2\Big).$$

Proof: The isometry property of multiple Wiener-Itô integrals and relation (3.16) imply

$$\begin{aligned}
\mathbf{E}(V^{3,N})^2 &= \frac{1}{N^2(\mathbf{E}(I_2(f_{i,N}))^3)^2}\sum_{i,j=0}^{N-1}\left[\mathbf{E}(I_2(g_{i,N})I_2(g_{j,N})) + 16\mathbf{E}(I_4(h_{i,N})I_4(h_{j,N}))\right.\\
&\qquad\qquad \left. + \mathbf{E}\left(I_6((f_{i,N}\tilde{\otimes}f_{i,N})\otimes f_{i,N})I_6((f_{j,N}\tilde{\otimes}f_{j,N})\otimes f_{j,N})\right)\right]\\
&= \frac{1}{C(H)^2N^{2-6H}}\left[\sum_{i,j=0}^{N-1} 2!\,\langle g_{i,N}, g_{j,N}\rangle_{L^2([0,1]^2)}\right.\\
&\qquad + \sum_{i,j=0}^{N-1} 4!\times 16\left\langle \widetilde{h_{i,N}}, \widetilde{h_{j,N}}\right\rangle_{L^2([0,1]^4)}\\
&\qquad \left. + \sum_{i,j=0}^{N-1} 6!\left\langle (f_{i,N}\tilde{\otimes}f_{i,N})\otimes f_{i,N}, (f_{j,N}\tilde{\otimes}f_{j,N})\otimes f_{j,N}\right\rangle_{L^2([0,1]^6)}\right]\\
&:= \frac{1}{C(H)^2N^{2-6H}}\left[A_N^{(2)} + B_N^{(4)} + D_N^{(6)}\right].
\end{aligned}$$

We use the notation $A_N^{(2)}$ to indicate that this term comes from the estimation of the second chaos summand of $V^{3,N}$, and similarly for the terms $B_N^{(4)}$ and $D_N^{(6)}$. We will try to estimate the all the three terms above to see which is the dominant term of V_N.

Estimation of the term $A_N^{(2)}$. We start by calculating $A_N^{(2)}$. Taking into account the expression of the second chaos kernel $g_{i,N}$ (3.12)

$$\begin{aligned}
A_N^{(2)} &:= \sum_{i,j=0}^{N-1} 2 < g_{i,N}, g_{j,N} >_{L^2([0,1]^2)}\\
&= 2\sum_{i,j=0}^{N-1}\Big[36\|f_{i,N}\|^2_{L^2([0,1]^2)}\|f_{j,N}\|^2_{L^2([0,1]^2)}\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)}\\
&\qquad +144\|f_{i,N}\|^2_{L^2([0,1]^2)}\langle f_{i,N}, (f_{j,N}\otimes_1 f_{j,N})\otimes_1 f_{j,N}\rangle_{L^2([0,1]^2)}\\
&\qquad +(24)^2\langle (f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N}, (f_{j,N}\otimes_1 f_{j,N})\otimes_1 f_{j,N}\rangle_{L^2([0,1]^2)}.\Big]\\
&:= 2(36A_{1,N}^{(2)} + 144A_{2,N}^{(2)} + (24)^2A_{3,N}^{(2)})
\end{aligned}$$

Let us evaluate the term $A_{1,N}^{(2)}$, we have

$$2!\|f_{i,N}\|^2_{L^2([0,1]^2)} = \mathbf{E}\left|Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right|^2 = N^{-2H}.$$

Furthermore

$$2\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)} = \mathbf{E}\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right)\left(Z^{(H)}_{\frac{j+1}{N}} - Z^{(H)}_{\frac{j}{N}}\right).$$

Hence

$$\begin{aligned} A^{(2)}_{1,N} &= \sum_{i,j=0}^{N-1}\left[\|f_{i,N}\|^2_{L^2([0,1]^2)}\|f_{j,N}\|^2_{L^2([0,1]^2)}\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)}\right] \\ &= \frac{1}{8}N^{-4H}\sum_{i,j=0}^{N-1}\mathbf{E}\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right)\left(Z^{(H)}_{\frac{j+1}{N}} - Z^{(H)}_{\frac{j}{N}}\right) = \frac{1}{8}N^{-4H} \end{aligned}$$

because $\mathbf{E}\sum_{i,j=0}^{N-1}\left(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}}\right)\left(Z^{(H)}_{\frac{j+1}{N}} - Z^{(H)}_{\frac{j}{N}}\right) = \mathbf{E}(Z^{(H)}_1)^2 = 1$ and we have

$$\lim_{N\to\infty} N^{4H}A^{(2)}_{1,N} = \frac{1}{8}. \tag{3.18}$$

We evaluate $A^{(2)}_{3,N}$. Note that

$$\begin{aligned} & d(H)^{-2}a(H)^{-1}(f_{i,N}\otimes_1 f_{i,N})(x,y) \\ = \; & 1^{\otimes 2}_{[0,\frac{i}{N}]}(x,y)\int_{I_i}\int_{I_i}\partial_1 K^{H'}(u_1,x)\partial_1 K^{H'}(u_2,y)|u_1-u_2|^{2H'-2}du_2du_1 \\ & +1_{[0,\frac{i}{N}]}(x)1_{I_i}(y)\int_{I_i}\int_y^{\frac{i+1}{N}}\partial_1 K^{H'}(u_1,x)\partial_1 K^{H'}(u_2,y)|u_1-u_2|^{2H'-2}du_2du_1 \\ & +1_{[0,\frac{i}{N}]}(y)1_{I_i}(x)\int_x^{\frac{i+1}{N}}\int_{I_i}\partial_1 K^{H'}(u_1,x)\partial_1 K^{H'}(u_2,y)|u_1-u_2|^{2H'-2}du_2du_1 \\ & +1_{I_i}(x)1_{I_i}(y)\int_x^{\frac{i+1}{N}}\int_y^{\frac{i+1}{N}}\partial_1 K^{H'}(u_1,x)\partial_1 K^{H'}(u_2,y)|u_1-u_2|^{2H'-2}du_2du_1. \end{aligned}$$

Sometimes its useful to use to following compressed expression

$$(f_{i,N}\otimes_1 f_{i,N})(x,y) = d(H)^2a(H)1^{\otimes 2}_{[0,\frac{i+1}{N}]}(x,y)\int_{I_i}\int_{I_i}\partial_1 K^{H'}(u_1,x)\partial_1 K^{H'}(u_2,y)|u_1-u_2|^{2H'-2}du_2du_1 \tag{3.19}$$

and

$$\begin{aligned} & d(H)^{-1}f_{i,N}(x,z) \\ = \; & 1^{\otimes 2}_{[0,\frac{i}{N}]}(x,z)\int_{I_i}\partial_1 K^{H'}(u_3,x)\partial_1 K^{H'}(u_3,z)du_3 + 1_{[0,\frac{i}{N}]}(x)1_{I_i}(z)\int_z^{\frac{i+1}{N}}\partial_1 K^{H'}(u_3,x)\partial_1 K^{H'}(u_3,z)du_3 \\ & +1_{[0,\frac{i}{N}]}(z)1_{I_i}(x)\int_x^{\frac{i+1}{N}}\partial_1 K^{H'}(u_3,x)\partial_1 K^{H'}(u_3,z)du_3 \\ & +1_{I_i}(x)1_{I_i}(z)\int_{I_i}\int_{I_i}\partial_1 K^{H'}(u_3,x)\partial_1 K^{H'}(u_3,z)du_3 \end{aligned}$$

or otherwise

$$f_{i,N}(x,z) = d(H)1^{\otimes 2}_{[0,\frac{i+1}{N}]}(x,z)\int_{I_i}\partial_1 K^{H'}(u_3,x)\partial_1 K^{H'}(u_3,z)du_3. \tag{3.20}$$

Therefore

$$((f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N})(y,z) = d(H)^3 a(H)^2 \Bigg(1^{\otimes 2}_{[0,\frac{i+1}{N}]}(y,z)$$
$$\times \int_{I_i}\int_{I_i}\int_{I_i} (|u_1-u_2||u_1-u_3|)^{2H'-2}\, \partial_1 K^{H'}(u_2,y)\partial_1 K^{H'}(u_3,z) du_3 du_2 du_1 \Bigg).$$

The norm has a nicer expression. Using the change of variables $\bar{u} = (u - \frac{i}{N})N$ (which is now usual and it can be used systematically) we have

$$\begin{aligned}
&\langle (f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}, (f_{j,N} \otimes_1 f_{j,N}) \otimes_1 f_{j,N} \rangle_{L^2([0,1]^2)} \\
= \;& d(H)^6 a(H)^6 \int_{I_i^3} du_1 du_2 du_3 \int_{I_j^3} dv_1 dv_2 dv_3 \\
& |u_1-u_2|^{2H'-2}|u_1-u_3|^{2H'-2}|v_1-v_2|^{2H'-2}|v_1-v_3|^{2H'-2}|u_2-v_2|^{2H'-2}||u_3-v_3|^{2H'-2} \\
= \;& \frac{d(H)^6 a(H)^6}{N^{12H'-6}} \int_{[0,1]^3}\int_{[0,1]^3} |v_1-v_2|^{2H'-2}|v_2-v_3|^{2H'-2}|v_3-v_4+i-j|^{2H'-2} \\
& \times |v_4-v_5|^{2H'-2}|v_5-v_6|^{2H'-2}|v_6-v_1+j-i|^{2H'-2} dv_1 \dots dv_6.
\end{aligned}$$

The rate of convergence of all terms presents in this proof comes actually from how many product $|u-v|^{2H'-2}$ with $u \in I_i$ and $v \in I_j$ we have. Hence

$$\begin{aligned}
A^{(2)}_{3,N} &= \frac{d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{i,j=1}^{N} \int_{[0,1]^3}\int_{[0,1]^3} |v_1-v_2|^{2H'-2}|v_2-v_3|^{2H'-2}|v_3-v_4+i-j|^{2H'-2} \\
&\qquad \times |v_4-v_5|^{2H'-2}|v_5-v_6|^{2H'-2}|v_6-v_1+j-i|^{2H'-2} dv_1 \dots dv_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{i>j\;=1}^{N} \int_{[0,1]^3}\int_{[0,1]^3} |v_1-v_2|^{2H'-2}|v_2-v_3|^{2H'-2}|v_3-v_4+i-j|^{2H'-2} \\
&\qquad \times |v_4-v_5|^{2H'-2}|v_5-v_6|^{2H'-2}|v_1-v_6+i-j|^{2H'-2} dv_1 \dots dv_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{k=0}^{N-1} (N-k) \int_{[0,1]^3}\int_{[0,1]^3} |v_1-v_2|^{2H'-2}|v_2-v_3|^{2H'-2}|v_3-v_4+k|^{2H'-2} \\
&\qquad \times |v_4-v_5|^{2H'-2}|v_5-v_6|^{2H'-2}|v_1-v_6+k|^{2H'-2} dv_1 \dots dv_6.
\end{aligned}$$

We put

$$\begin{aligned}
\overline{A}^{(2)}_{3,N} &:= \frac{1}{N^{12H'-6}} \sum_{k=0}^{N-1} (N-k)|v_3-v_4+k|^{2H'-2}|v_1-v_6+k|^{2H'-2} \\
&= \frac{1}{N^{8H'-4}} \frac{1}{N} \sum_{k=0}^{N-1} (1-\frac{k}{N}) |\frac{v_3-v_4}{N} + \frac{k}{N}|^{2H'-2} |\frac{v_1-v_6}{N} + \frac{k}{N}|^{2H'-2}
\end{aligned}$$

and we conclude easily by a Riemann sum convergence that

$$N^{4H}\overline{A}_{3,N} = N^{8H'-4}\overline{A}_{3,N} \underset{N\to\infty}{\longrightarrow} \int_0^1 (1-x)x^{4H'-4} dx = \frac{1}{2H-1} - \frac{1}{2H}$$

because the terms $\frac{v_3-v_4}{N}$ are negligible with respect to $\frac{k}{N}$ for large enough k. This implies that,

$$N^{4H}A^{(2)}_{3,N} \underset{N\to\infty}{\longrightarrow} \frac{d(H)^6a(H)^6}{H(2H-1)}(C'(H))^2 = \frac{H^2(2H-1)^2}{8}(C'(H))^2 \tag{3.21}$$

where

$$C'(H) = \int_{[0,1]^3} |v_1-v_2|^{2H'-2}|v_2-v_3|^{2H'-2}dv_1dv_2dv_3.$$

Now, we estimate the term $A^{(2)}_{2,N}$.

$$\begin{aligned}
&\langle (f_{i,N}, (f_{j,N}\otimes_1 f_{j,N})\otimes_1 f_{j,N}\rangle_{L^2([0,1]^2)} \\
=\;& d(H)^4a(H)^4\int_{I_i}\int_{I_j^3}|u_1-u_2|^{2H'-2}|u_2-u_3|^{2H'-2}|u_3-u_4|^{2H'-2}|u_4-u_1|^{2H'-2}du_1\dots du_4 \\
=\;& \frac{d(H)^4a(H)^4}{N^{8H'-4}}\int_{[0,1]^4}|v_1-v_2+i-j|^{2H'-2}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2}|v_4-v_1+j-i|^{2H'-2}dv_1\dots dv_4
\end{aligned}$$

Then

$$\begin{aligned}
A^{(2)}_{2,N} &= \|f_{i,N}\|^2_{L^2([0,1]^2)}\frac{d(H)^4a(H)^4}{N^{8H'-4}}\int_{[0,1]^4}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2} \\
&\qquad \times\left(\sum_{i,j=1}^{N}|v_1-v_2+i-j|^{2H'-2}|v_4-v_1+j-i|^{2H'-2}\right)dv_1\dots dv_4 \\
&= \frac{N^{-2H}}{2}\frac{d(H)^4a(H)^4}{N^{8H'-4}}\int_{[0,1]^4}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2} \\
&\qquad \times\left(2\sum_{k=0}^{N-1}(N-k)|v_1-v_2+k|^{2H'-2}|v_4-v_1+k|^{2H'-2}\right)dv_1\dots dv_4 \\
&= N^{-2H}\frac{d(H)^4a(H)^4}{N^{4H'-2}}\int_{[0,1]^4}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2} \\
&\qquad \times\left(\frac{1}{N}\sum_{k=0}^{N-1}(1-\frac{k}{N})|\frac{v_1-v_2}{N}+\frac{k}{N}|^{2H'-2}|\frac{v_4-v_1}{N}+\frac{k}{N}|^{2H'-2}\right)dv_1\dots dv_4 \\
&= \frac{d(H)^4a(H)^4}{N^{4H}}\int_{[0,1]^4}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2} \\
&\qquad \times\left(\frac{1}{N}\sum_{k=0}^{N-1}(1-\frac{k}{N})|\frac{v_1-v_2}{N}+\frac{k}{N}|^{2H'-2}|\frac{v_4-v_1}{N}+\frac{k}{N}|^{2H'-2}\right)dv_1\dots dv_4.
\end{aligned}$$

We obtain that $N^{4H}A^{(2)}_{2,N}$ converges as $N\to\infty$ to

$$\begin{aligned}
&(\int_0^1(1-x)x^{4H'-4}dx)\left(d(H)^4a(H)^4\int_{[0,1]^3}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2}dv_2\dots dv_4\right) \\
=\;& (\frac{1}{2H-1}-\frac{1}{2H})\left(d(H)^4a(H)^4\int_{[0,1]^3}|v_2-v_3|^{2H'-2}|v_3-v_4|^{2H'-2}dv_2\dots dv_4\right)
\end{aligned}$$

Thus

$$N^{4H} A_{2,N}^{(2)} \underset{N\to\infty}{\longrightarrow} C'(H)\frac{H(2H-1)}{8} \tag{3.22}$$

From (3.18), (3.21) and (3.22), we obtain that

$$N^{4H} A_N^{(2)} = 2(36N^{4H}A_{1,N}^{(2)} + 144N^{4H}A_{2,N}^{(2)} + (24)^2 N^{4H} A_{3,N}^{(2)}) \tag{3.23}$$

converges to $\left(9 + 36C'(H)H(2H-1) + 144\left[C'(H)H(2H-1)\right]^2\right) := C_0(H)$ as $N \longrightarrow \infty$. Consequently

$$\frac{N^{4H}A_N^{(2)}}{C_0(H)} \underset{N\to\infty}{\longrightarrow} 1. \tag{3.24}$$

Estimation of the term $D_N^{(6)}$. Now, we study the convergence of $D_N^{(6)}$, using the symmetry property of every $f_{i,N},\ i=0,\dots,N-1$ on $[0,1]^2$, there exist positive combinatorial constants c_1, c_2 and c_3 such that

$$\begin{aligned}
D_N^{(6)} &= \sum_{i,j=0}^{N-1} 6!\left\langle (f_{i,N}\tilde{\otimes} f_{j,N})\tilde{\otimes} f_{j,N}), (f_{j,N}\tilde{\otimes} f_{i,N})\tilde{\otimes} f_{i,N})\right\rangle_{L^2([0,1]^6)} \\
&= c_1 \sum_{i,j=0}^{N-1} \left(\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)}\right)^3 \\
&+ c_2 \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)} \int_{[0,1]^4} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_1) dx_1\dots dx_4 \\
&+ c_3 \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_5) f_{i,N}(x_5,x_6) f_{j,N}(x_6,x_1) dx_1\dots dx_6 \\
&:= c_1 D_{1,N}^{(6)} + c_2 D_{2,N}^{(6)} + c_3 D_{3,N}^{(6)}.
\end{aligned}$$

By using the same argument as above, we have

$$\begin{aligned}
D_{1,N}^{(6)} &= \sum_{i,j=0}^{N-1} \left(\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)}\right)^3 \\
&= \frac{d(H)^6 a(H)^6}{N^{6H}} \sum_{i,j=0}^{N-1} \left(\int_{[0,1]^2} |x_1 - x_2 + i - j|^{2H-2}\, d_1 dx_2\right)^3 \\
&= \frac{d(H)^6 a(H)^6}{N^{6H}} 2\sum_{k=0}^{N-1} (N-k) \left(\int_{[0,1]^2} |x_1 - x_2 + k|^{2H-2}\, dx_1 dx_2\right)^3 \\
&= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N}(1-\frac{k}{N}) \left(\int_{[0,1]^2} \left|\frac{x_1-x_2}{N} + \frac{k}{N}\right|^{2H-2}\right)^3.
\end{aligned}$$

and clearly since $H < 1$ we have

$$\lim_{N\to\infty} N^{4H} D_{1,N}^{(6)} = 0. \tag{3.25}$$

By the same way, we obtain

$$\begin{aligned}
D_{2,N}^{(6)} &= \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)} \int_{[0,1]^4} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_1) dx_1 \dots dx_4 \\
&= \sum_{i,j=0}^{N-1} N^{-4} d(H)^4 a(H)^4 \left(\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)}\right) \int_{[0,1]^4} \left|\frac{x_1-x_2}{N} + \frac{i-j}{N}\right|^{2H'-2} \\
&\qquad \times \left|\frac{x_3-x_2}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_3-x_4}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_1-x_4}{N} + \frac{i-j}{N}\right|^{2H'-2} dx_1 \dots dx_4 \\
&= \frac{2d(H)^6 a(H)^6}{N^6} \sum_{k=0}^{N-1} (N-k) \int_{[0,1]^2} \left|\frac{x_5-x_6}{N} + \frac{k}{N}\right|^{2H-2} dx_5 dx_6 \int_{[0,1]^4} \left|\frac{x_1-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \\
&\quad \times \left|\frac{x_3-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_3-x_4}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_1-x_4}{N} + \frac{k}{N}\right|^{2H'-2} dx_1 \dots dx_4 \\
&= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N}(1-\frac{k}{N}) \int_{[0,1]^2} \left|\frac{x_5-x_6}{N} + \frac{k}{N}\right|^{2H-2} dx_5 dx_6 \int_{[0,1]^4} \left|\frac{x_1-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \\
&\quad \times \left|\frac{x_3-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_3-x_4}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_1-x_4}{N} + \frac{k}{N}\right|^{2H'-2} dx_1 \dots dx_4.
\end{aligned}$$

This implies that

$$\lim_{N\to\infty} N^{4H} D_{2,N}^{(6)} = 0. \tag{3.26}$$

The same manner as in previous results, we have

$$\begin{aligned}
D_{3,N}^{(6)} &= \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_5) f_{i,N}(x_5,x_6) f_{j,N}(x_6,x_1) dx_1 \dots dx_6 \\
&= \frac{d(H)^6 a(H)^6}{N^6} \sum_{i,j=0}^{N-1} \int_{[0,1]^6} \left|\frac{x_1-x_2}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_3-x_2}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_3-x_4}{N} + \frac{i-j}{N}\right|^{2H'-2} \\
&\qquad \times \left|\frac{x_5-x_4}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_5-x_6}{N} + \frac{i-j}{N}\right|^{2H'-2} \left|\frac{x_1-x_6}{N} + \frac{i-j}{N}\right|^{2H'-2} dx_1 \dots dx_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N}(1-\frac{k}{N}) \int_{[0,1]^4} \left|\frac{x_1-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_3-x_2}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_3-x_4}{N} + \frac{k}{N}\right|^{2H'-2} \\
&\quad \times \left|\frac{x_5-x_4}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_5-x_6}{N} + \frac{k}{N}\right|^{2H'-2} \left|\frac{x_1-x_6}{N} + \frac{k}{N}\right|^{2H'-2} dx_1 \dots dx_6.
\end{aligned}$$

Hence

$$\lim_{N\to\infty} N^{4H} D_{3,N}^{(6)} = 0. \tag{3.27}$$

Thus, from (3.25), (3.26) and (3.27), we obtain

$$\lim_{N\to\infty} N^{4H} D_N^{(6)} = 0. \tag{3.28}$$

Estimation of the term $B_N^{(4)}$. Applying the same argument as in last part, there exist constants c_1' and c_2' such that

$$\begin{aligned}
&\left\langle \widetilde{h_{i,N}}, \widetilde{h_{j,N}} \right\rangle_{L^2([0,1]^4)} \\
= \ & c_1' \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^2)} \int_{[0,1]^4} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_1) dx_i \\
+ \ & c_2' \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1,x_2) f_{j,N}(x_2,x_3) f_{i,N}(x_3,x_4) f_{j,N}(x_4,x_5) f_{i,N}(x_5,x_6) f_{j,N}(x_6,x_1) dx_i \\
= \ & c_1' D_{2,N}^{(6)} + c_2' D_{3,N}^{(6)}.
\end{aligned}$$

The same terms as in the estimation of the sixth chaos kernel appear. Thus, from the convergences (3.26) and (3.27),

$$\begin{aligned}
\lim_{N\to\infty} N^{4H} B_N^{(4)} &= \lim_{N\to\infty} N^{4H} \sum_{i,j=0}^{N-1} \left\langle \widetilde{h_{i,N}}, \widetilde{h_{j,N}} \right\rangle_{L^2([0,1]^4)} \\
&= \lim_{N\to\infty} \left(c_1' N^{4H} D_{2,N}^{(6)} + c_2' N^{4H} D_{3,N}^{(6)} \right) = 0. \qquad (3.29)
\end{aligned}$$

As a consequence of the convergences (3.24), (3.28) and (3.29), we have proved that for every $H > \frac{1}{2}$ and with the notation $\bar{C}(H) = \frac{C(H)^2}{C_0(H)}$,

$$\frac{C(H)^2}{C_0(H)} N^{2-2H} \mathbf{E}(V^{3,N})^2 = \mathbf{E}\left(\sqrt{\bar{C}(H)} N^{1-H} V^{3,N} \right)^2 \underset{N\to\infty}{\longrightarrow} 1. \tag{3.30}$$

3.3.2 Non-convergence to a Gaussian limit

We prove that the normalized variation doesn't converge in distribution to the normal law. Of course this is somehow superflu taking into account that in the next section we show a non-central limit theorem for this statistics, but we found the calculations instructive to see why it does not converges to a Gaussian limit. Recall that by a result of [83] (Theorem 4 in this paper) a sequence $F_N = I_q(f_N)$ in the qth Wiener chaos with $\mathbb{E}F_N^2 \underset{N\to\infty}{\longrightarrow} 1$ converges to the normal law $N(0,1)$ if and only if $\|DF_N\|^2_{L^2[0,1]}$ converges to q in $L^2(\Omega)$ when $N \to \infty$. Here D denotes the Malliavin derivative and if $f \in L^2([0,T]^n)$ is a symmetric function, we will use the following rule to differentiate in the Malliavin sense

$$D_t I_n(f) = n\, I_{n-1}(f(\cdot,t)), \quad t \in [0,1].$$

We put

$$T_N = \frac{\sqrt{\bar{C}(H)}N^{1-H}}{C(H)N^{1-3H}} \sum_{i=0}^{N} I_2(g_{i,N}) = \frac{N^{2H}}{\sqrt{C_0(H)}} \sum_{i=0}^{N} I_2(g_{i,N})$$

We derive T_N in the Malliavin sense and we obtain $D_tT_N = \frac{2N^{2H}}{\sqrt{C_0(H)}} \sum_{i=0}^{N} I_1(g_{i,N}(.,t))$ and thus

$$\begin{aligned}
\|DT_N\|^2_{L^2([0,1]^2)} &= \frac{4N^{4H}}{C_0(H)} \int_0^1 \left(\sum_{i=0}^{N} I_1(g_{i,N}(.,t)) \right)^2 dt \\
&= \frac{4N^{4H}}{C_0(H)} \int_0^1 \left(\sum_{i,j=0}^{N} I_1(g_{i,N}(.,t)) I_1(g_{j,N}(.,t)) \right) dt \\
&= \frac{4N^{4H}}{C_0(H)} \left(\sum_{i,j=0}^{N} \int_0^1 I_0(g_{i,N} \otimes_1 g_{j,N})dt + \sum_{i,j=0}^{N} \int_0^1 I_2(g_{i,N} \otimes_0 g_{j,N})dt \right) \\
&:= \frac{4N^{4H}}{C_0(H)} (J_{1,N} + J_{2,N})
\end{aligned}$$

where we denoted

$$J_{1,N} = \sum_{i,j=0}^{N} \int_0^1 I_0(g_{i,N} \otimes_1 g_{j,N})dt = \sum_{i,j=0}^{N} < g_{i,N}, g_{j,N} >_{L^2([0,1]^2)} = \frac{1}{2} A_N^{(2)}.$$

From (3.24) we obtain

$$\frac{4N^{4H}}{C_0(H)} J_{1,N} \underset{N \to \infty}{\longrightarrow} 2 \tag{3.31}$$

in $L^2(\Omega)$ because the term $A_N^{(2)}$ is determinist. To prove that $\|DT_N\|^2_{L^2([0,1]^2)}$ not converges in $L^2(\Omega)$ to 2, it is sufficient to show that

$$\lim_{N \to \infty} \mathbf{E} \left(\frac{4N^{4H}}{C_0(H)} J_{2,N} \right)^2 > 0.$$

where $J_{2,N} = \sum_{i,j=0}^{N} \int_0^1 I_2(g_{i,N}(.,t) g_{j,N}(.,t))dt$.

We calculate the mean square of this term.

$$\begin{aligned}
\mathbf{E}(J_{2,N})^2 &= 2\int_{[0,1]^2}\left(\sum_{i,j=0}^{N}\int_0^1 I_2(g_{i,N}(r,t)g_{j,N}(s,t))dt\right)^2 drds \\
&= 2\sum_{i,j,k,l=0}^{N}\int_{[0,1]^4} g_{i,N}(r,t)g_{j,N}(s,t)g_{k,N}(r,u)g_{l,N}(s,u)drdsdtdu \\
&\geq \sum_{i,j,k,l=0}^{N}\int_{[0,1]^4} f_{i,N}(r,t)f_{j,N}(s,t)f_{k,N}(r,u)f_{l,N}(s,u)drdsdtdu \\
&= 2d(H)^4a(H)^4N^{-8H}\sum_{i,j,k,l=0}^{N}\int_{I_i}\int_{I_j}\int_{I_k}\int_{I_l}(|r-t||s-t||r-u||s-u|)^{2H'-2}drdsdtdu \\
&= 2d(H)^4a(H)^4N^{-8H}\sum_{i,j,k,l=0}^{N}\int_{[0,1]^4} drdsdtdu \\
&\quad \frac{1}{N^4}\left(|\frac{r-t+i-j}{N}||\frac{t-s+j-k}{N}||\frac{s-u+k-l}{N}||\frac{u-r+l-i}{N}|\right)^{2H'-2}.
\end{aligned}$$

By using Riemann sums approximations, we obtain

$$\lim_{N\to\infty}\mathbf{E}(N^{4H}J_{2,N})^2 \geq 2d(H)^4a(H)^4\int_{[0,1]^4} dx_1dx_2dx_3dx_4\,(|x_1-x_2||x_2-x_3||x_3-x_4||x_4-x_1|)^{2H'-2} > 0.$$

3.4 The non-central limit theorem for the cubic variation of the Rosenblatt process

Denote by L_t the kernel of the Rosenblatt process

$$L_t(x,y) = d(H)1_{[0,1]^{\otimes 2}}(x,y)\int_0^t\int_0^t\left(\int_{x\vee y}^t \partial_1 K^{H'}(s,x)\partial_1 K^{H'}(s,y)ds\right)$$

and recall the notation

$$f_{i,N}(x,y) = L^{(H)}_{\frac{i+1}{N}}(x,y) - L^{(H)}_{\frac{i}{N}}(x,y).$$

We proved in the previous section that the dominant term of the statistics $V^{3,N}$ which gives its normalization is

$$C(H)^{-1}N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N})$$

where

$$\begin{aligned}
g_{i,N} &= 6\|f_{i,N}\|^2_{L^2([0,1]^2)}f_{i,N} + 24\,(f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N} \\
&= 3N^{-2H}f_{i,N} + 24\,(f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N} := 3g^{(1)}_{i,N} + 24g^{(2)}_{i,N}.
\end{aligned}$$

More precisely, it follows from the proof of Proposition 2 that

$$\mathbf{E}\left[N^{1-H}\left(N^{3H-1}\sum_{i=0}^{N-1} I_2(g^{(1)}_{i,N})\right)\right]^2 = N^{4H}A^{(2)}_{1,N} \to_{N\to\infty} 1/8$$

and

$$\mathbf{E}\left[N^{1-H}\left(N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)})\right)\right]^2 = N^{4H}A_{3,N}^{(2)} \to_{N\to\infty} \frac{H^2(2H-1)^2}{8}(C'(H))^2.$$

Consequently, the limit of the sequence $V_{3,N}$ is the same as the limit of the sequence

$$C(H)^{-1}N^{1-H}N^{3H-1}\left(3\sum_{i=0}^{N-1} I_2\left(g_{i,N}^{(1)}\right) + 24\sum_{i=0}^{N-1} I_2\left(g_{i,N}^{(2)}\right)\right).$$

We prove here our main result.

Theorem 9. *The renormalized cubic variation statistics based on the Rosenblatt process $N^{1-H}V^{3,N}$ with $V^{3,N}$ given by (3.10) converges in $L^2(\Omega)$ as $N \to \infty$ to the Rosenblatt random variable $D(H)Z_1^{(H)}$ where $D(H) = C(H)^{-1}(3 + 24d(H)^2a(H)^2C'(H))$.*

Proof: To see the limit of $N^{1-H}V^{3,N}$ we need therefore to study the convergence of $N^{1-H}\left(N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)})\right)$ and of $N^{1-H}\left(N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)})\right)$.

Is easy to treat the first part. In fact we have

$$N^{1-H}N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)}) = N^{2H}\sum_{i=0}^{N-1} N^{-2H}I_2(f_{i,N}) = \sum_{i=0}^{N-1} I_2(f_{i,N}) = Z_1^H \tag{3.32}$$

where Z_1^H is a Rosenblatt random variable with selfsimilarity order H.

We find then the limit of the second part of the dominant term. We have

$$N^{1-H}N^{3H-1}\sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)}) = N^{2H}\sum_{i=0}^{N-1} I_2\left((f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N}\right).$$

Let us denote by

$$l^{H'}(x,y,z,t) := \partial_1 K^{H'}(x,y)\partial_1 K^{H'}(z,t)$$

and by

$$\begin{aligned} l_0^{H'}(x,y,z) &:= \partial_1 K^{H'}(x,y)\partial_1 K^{H'}(x,z) = l^{H'}(x,y,x,t)\\ l_1^{H'}(x,y,z) &:= \partial_1 K^{H'}(x,z)\partial_1 K^{H'}(y,z) = l^{H'}(x,z,y,z). \end{aligned}$$

Using the relations (3.19) and (3.20) we get

$$\begin{aligned} & ((f_{i,N}\otimes_1 f_{i,N})\otimes_1 f_{i,N})(y_1,y_2)\\ = \; & d(H)^3a(H)^2 1_{[0,\frac{i}{N}]}^{\otimes 2}(y_1,y_2)\int_{I_i^3} du_1du_2du_3 l^{H'}(u_1,y_1,u_3,y_2)\left[|u_1-u_2||u_2-u_3|\right]^{2H'-2}\\ := \; & b_{i,N}^{(1)}(y_1,y_2) + b_{i,N}^{(2)}(y_1,y_2) \end{aligned}$$

with

$$b_{i,N}^{(1)}(y_1,y_2) = d(H)^3 a(H)^2 1_{[0,\frac{i}{N}]}^{\otimes 2}(y_1,y_2)\int_{I_i^3} du_1du_2du_3 l^{H'}(u_1,y_1,u_3,y_2)\left[|u_1-u_2||u_2-u_3|\right]^{2H'-2}$$

and

$$\begin{aligned} b_{i,N}^{(2)}(y_1,y_2) &= d(H)^3a(H)^2 \\ &\times \Bigg[1_{I_i}(y_1)1_{[0,\frac{i}{N}]}(y_2)\int_{y_1}^{\frac{i+1}{N}} du_1\int_{I_i^2} du_2du_3 l^{H'}(u_1,y_1,u_3,y_2)\left[|u_1-u_2||u_2-u_3|\right]^{2H'-2} \\ &+ 1_{[0,\frac{i}{N}]}(y_1)1_{I_i}(y_2)\int_{I_i^2} du_1du_2\int_{y_2}^{\frac{i+1}{N}} du_3 l^{H'}(u_1,y_1,u_3,y_2)\left[|u_1-u_2||u_2-u_3|\right]^{2H'-2} \\ &+ 1_{I_i^2}(y_1,y_2)\int_{y_1}^{\frac{i+1}{N}} du_1\int_{I_i} du_2\int_{y_2}^{\frac{i+1}{N}} du_3 l^{H'}(u_1,y_1,u_3,y_2)\left[|u_1-u_2||u_2-u_3|\right]^{2H'-2}\Bigg]. \end{aligned}$$

We show that the $I_2(N^{2H}\sum_{i=0}^{N-1} b_{i,N}^{(2)})$ converges to zero in $L^2(\Omega)$ and it has no contribution to the limit. Indeed,

$$\begin{aligned} &\mathbf{E}\left(I_2(N^{2H}\sum_{i=0}^{N-1} b_{i,N}^{(2)})\right)^2 = 2N^{4H}\int_0^1\int_0^1 dy_1dy_2\left(\sum_{i=0}^{N-1} b_{i,N}^2(y_1,y_2))\right)^2 \\ \leq\ & 2d(H)^6a(H)^4N^{4H}\Bigg(\sum_{i=0}^{N-1}\int_0^1\int_0^1 dy_1dy_2\int_{I_i^6} du_1du_1'du_2du_2'du_3du_3' 1_{[y_1,\frac{i+1}{N}]^2}(u_1,u_1') \\ \times\ & 1_{[y_2,\frac{i+1}{N}]^2}(u_3,u_3') l_1^{H'}(u_1,u_1',y_1) l_1^{H'}(u_3,u_3',y_2)\left[|u_1-u_2||u_2-u_3||u_1'-u_2'||u_2'-u_3'|\right]^{2H'-2}\Bigg) \\ \leq\ & 2d(H)^6a(H)^4N^{4H}\Bigg(\sum_{i=0}^{N-1}\int_{I_i^6} du_1du_1'du_2du_2'du_3du_3'\left[|u_1-u_2||u_2-u_3||u_1'-u_2'||u_2'-u_3'|\right]^{2H'-2} \\ \times\ & \int_0^{u_1\wedge u_1'} dy_1 l_1^{H'}(u_1,u_1',y_1)\int_0^{u_3\wedge u_3'} dy_2 l_1^{H'}(u_3,u_3',y_2)\Bigg) \\ \leq\ & 2d(H)^6a(H)^4N^{4H}\Bigg(\sum_{i=0}^{N-1}\int_{I_i^6} du_1du_1'du_2du_2'du_3du_3' \\ \times\ & \left[|u_1-u_2||u_2-u_3||u_1'-u_2'||u_2'-u_3'||u_1-u_1'||u_1-u_3'|\right]^{2H'-2}\Bigg) \\ \leq\ & 2d(H)^6a(H)^4N^{4H}NN^{-6}N^{12-12H'}\Bigg(\int_{[0,1]^6} dv_1dv_1'dv_2dv_2'dv_3dv_3' \\ \times\ & \left[|v_1-v_2||v_2-v_3||v_1'-v_2'||v_2'-v_3'||v_1-v_1'||v_1-v_3'|\right]^{2H'-2}\Bigg) \\ \leq\ & 2cd(H)^6a(H)^4N^{1-2H}. \end{aligned}$$

Combining with the fact that $H > \frac{1}{2}$, we conclude that

$$I_2(N^{2H}\sum_{i=0}^{N-1} b_{i,N}^{(2)}) \underset{N\to\infty}{\longrightarrow} 0 \qquad \text{in } L^2(\Omega), \tag{3.33}$$

and then we need to find the limit of

$$\begin{aligned} N^{2H}\sum_{i=0}^{N-1} b_{i,N}^{(1)} &= d(H)^3 a(H)^2 N^{2H}\sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]^2}(y_1,y_2) \\ &\quad \times \int_{I_i^3} du_1du_2du_3 l^{H'}(u_1,y_1,u_3,y_2)\,[|u_1-u_2||u_2-u_3|]^{2H'-2} \\ &= d(H)^3\,(a(H))^2 \sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]^2}(y_1,y_2)N^{-1} \\ &\quad \times \int_{[0,1]^3} dv_1dv_2dv_3 l^{H'}(\frac{v_1+i}{N},y_1,\frac{v_3+i}{N},y_2)\,[|v_1-v_2||v_2-v_3|]^{2H'-2}. \end{aligned}$$

The last sequence has the same limit pointwise (for every y_1,y_2) as

$$\begin{aligned} d(H)^3 a(H)^2 &\int_{[0,1]^3} dv_1dv_2dv_3\,[|v_1-v_2||v_2-v_3|]^{2H'-2} \\ &\times \sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]^2}(y_1,y_2)N^{-1} l^{H'}(\frac{i}{N},y_1,\frac{i}{N},y_2). \end{aligned} \tag{3.34}$$

This last term is a Riemann sum that converges to

$$\begin{aligned} & d(H)^3 a(H)^2 \int_{[0,1]^3} dv_1dv_2dv_3\,[|v_1-v_2||v_2-v_3|]^{2H'-2}\int_{y_1\vee y_2}^1 dx l^{H'}(x,y_1,x,y_2) \\ =\; & d(H)^3 a(H)^2 C'(H)\int_{y_1\vee y_2}^1 dx\partial_1 K^{H'}(x,y_1)\partial_1 K^{H'}(x,y_2) = d(H)^2 a(H)^2 C'(H) L_1^{(H)}(x,y) \end{aligned}$$

where $L_1^{(H)}$ is the standard kernel of the Rosenblatt process (3.6).

We need a Cauchy sequence argument as in [112] to conclude the proof. That is, we will show that the sequence $N^{2H}\sum_{i=0}^{N-1} b_{i,N}^{(1)}$ (or equivalently $N^{2H}\sum_{i=0}^{N-1} g_{i,N}^{(2)}$) is Cauchy in the Hilbert space $L^2([0,1]^2)$. This will imply that the sequence of random variable $I_2\left(N^{2H}\sum_{i=0}^{N-1} g_{i,N}^{(2)}\right)$ is Cauchy, so convergent, in the space $L^2(\Omega)$ and we deduce as in [112] that its limit coincides with the multiple integral of the pointwise limit of the kernel. We compute, for $M, N \geq 1$

$$
\begin{aligned}
&\left\| N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)} - M^{2H} \sum_{i=0}^{M-1} b_{i,M}^{(1)} \right\|_{L^2([0,1]^2)}^2 \\
= \;& d(H)^6 a(H)^4 \left[N^{4H} \sum_{i,j=0}^{N-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'| \right]^{2H'-2} \right. \\
& \times \int_0^{u_1 \wedge u_1'} dy_1 \int_0^{u_3 \wedge u_3'} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u_1', y_1, u_3', y_2) \\
+ \;& M^{4H} \sum_{i,j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'| \right]^{2H'-2} \\
& \times \int_0^{u_1 \wedge u_1'} dy_1 \int_0^{u_3 \wedge u_3'} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u_1', y_1, u_3', y_2) \\
- \;& 2N^{2H} M^{2H} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'| \right]^{2H'-2} \\
& \left. \times \int_0^{u_1 \wedge u_1'} dy_1 \int_0^{u_3 \wedge u_3'} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u_1', y_1, u_3', y_2) \right] \\
= \;& d(H)^6 a(H)^4 \left[N^{4H} \sum_{i,j=0}^{N-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \right. \\
& \times \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'||u_1 - u_1'||u_3 - u_3'| \right]^{2H'-2} \\
+ \;& M^{4H} \sum_{i,j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \\
& \times \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'||u_1 - u_1'||u_3 - u_3'| \right]^{2H'-2} \\
- \;& 2N^{2H} M^{2H} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du_1' du_2' du_3' \\
& \left. \times \left[|u_1 - u_2||u_2 - u_3||u_1' - u_2'||u_2' - u_3'||u_1 - u_1'||u_3 - u_3'| \right]^{2H'-2} \right]
\end{aligned}
$$

and this equal to

$$\begin{aligned}
& d(H)^6 a(H)^4 \Bigg[N^{-2H} \sum_{i,j=0}^{N-1} \int_{[0,1]^6} du_1 du_2 du_3 du_1' du_2' du_3' \\
& \times \left[|v_1 - v_2||v_2 - v_3||v_1' - v_2'||v_2' - v_3'||v_1 - v_1' + i - j||v_3 - v_3' + i - j| \right]^{2H'-2} \\
+ \quad & M^{-2H} \sum_{i,j=0}^{M-1} \int_{[0,1]^6} du_1 du_2 du_3 du_1' du_2' du_3' \\
& \times \left[|v_1 - v_2||v_2 - v_3||v_1' - v_2'||v_2' - v_3'||v_1 - v_1' + i - j||v_3 - v_3' + i - j| \right]^{2H'-2} \\
- \quad & 2N^{-1}M^{-1} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{[0,1]^6} du_1 du_2 du_3 du_1' du_2' du_3' \\
& \times \left[|v_1 - v_2||v_2 - v_3||v_1' - v_2'||v_2' - v_3'|| \frac{v_1}{N} - \frac{v_1'}{M} + \frac{i}{N} - \frac{j}{M} || \frac{v_3}{N} - \frac{v_3'}{M} + \frac{i}{N} - \frac{j}{M} | \right]^{2H'-2} \Bigg]
\end{aligned}$$

The same way as in above this las term when $N \longrightarrow \infty$ and $N \longrightarrow \infty$ converges to

$$\begin{aligned}
& d(H)^6 a(H)^4 \left[2 \int_0^1 (1-x) x^{2H-2} dx + 2 \int_0^1 (1-x) x^{2H-2} dy + 2 \int_0^1 \int_0^1 |x-y|^{2H-2} dx dy \right] \\
& = \frac{1}{H(2H-1)} + \frac{1}{H(2H-1)} - \frac{2}{H(2H-1)} = 0.
\end{aligned}$$

We obtained that $\{N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)}, N \geq 0\}$ is a Cauchy sequence and this completes the proof.

Chapter 4

Estimation of the drift of fractional Brownian motion

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := \left(B_t^H\right)_{t\in[0,T]}$ with hurst parameter H less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

4.1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \left\{(B_t^{H,1}, ..., B_t^{H,d}); t \in [0,T]\right\}$ be a d-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, B^H is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1, ..., d$ $B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B_s^{H,i} B_t^{H,i}) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s, t \in [0,T].$$

For each $i = 1, \dots, d$, $\left(\mathcal{F}_t^i\right)_{t\in[0,T]}$ denotes the filtration generated by $\left(B_t^{H,i}\right)_{t\in[0,T]}$.
The fBm was first introduced by Kolmogorov [60] and studied by Mandelbrot and Van Ness in [71]. Notice that if $H = \frac{1}{2}$, the process $B^{\frac{1}{2}}$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let M be a subspace of the Cameron-Martin space defined by

$$M = \left\{\varphi : [0,T] \to \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ with } \dot{\varphi}^i \in L^2([0,T]) \right.$$
$$\left. \text{and } \varphi^i \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T])\right), i = 1, ..., d\right\}.$$

Let $\theta = \left\{(\theta_t^1, \dots, \theta_t^d); t \in [0,T]\right\}$ be a function belonging to M. Then, Applying Girsanov theorem (see Theorem 2 in [84]), there exist a probability measure absolutely continuous with respect to P under which the process $\widetilde{B}^H$ defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0,T] \tag{4.1}$$

is a fBm with Hurst parameter H and mean zero. In this case, we say that, under the probability P_θ, the process B^H is a fBm with drift θ.

We consider in this paper the problem of estimating the drift θ of B^H under the probability P_θ, with hurst parameter $H < 1/2$. We wish to estimate θ under the usual quadratic risk, that is defined for any estimator δ of θ by

$$\mathcal{R}(\theta, \delta) = E_\theta \left[\int_0^T ||\delta_t - \theta_t||^2 dt \right]$$

where E_θ is the expectation with respect to a probability P_θ.

Let $X = (X^1, \ldots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of θ is X itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by X. That is

$$\sigma^2 d = E\left[\|X - \theta\|_d^2\right] = \inf_{\xi \in \mathcal{S}} E\left[\|\xi - \theta\|_d^2\right],$$

where $\mathcal{S}$ is the class of unbiased estimators of θ and $\|.\|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

Stein [107] constructed biased superefficient estimators of θ of the form

$$\delta_{a,b}(X) = \left(1 - \frac{b}{a + ||X||^2}\right) X$$

for a sufficiently small and b sufficiently large when $d \geq 3$. James and Stein [57] sharpened later this result and presented an explicit class of biased superefficient estimators of the form

$$\left(1 - \frac{a}{||X||_d^2}\right) X, \text{ for } 0 < a < 2(d-2).$$

Recently, an infinite-dimensional extension of this result has been given by Privault and Reveillac in [96]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t\in[0,T]}$ of a continuous Gaussian martingale $(X_t)_{t\in[0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t\in[0,T]}$ is an efficient estimator of $(\theta_t)_{t\in[0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian processe of the form:

$$X_t := \int_0^t K(t,s) dW_s, \qquad t \in [0,T],$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion and $K(.,.)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where F is a positive superharmonic random variable and D is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\widehat{\theta} = B^H$ is an efficient estimator of θ under the probability P_θ with risk

$$\mathcal{R}(\theta, B^H) = E_\theta \left[\int_0^T \|B_t^H - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1} d.$$

Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ.

In Section 4, we construct a class of biased estimators of James-Stein type of the form

$$\delta(B^H)_t = \left(1 - at^{2H}\left(\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right)\right) B_t^H, \qquad t \in [0,T].$$

We give sufficient conditions on the function r and on the constant a in order that $\delta(B^H)$ dominates B^H under the usual quadratic risk i.e.

$$\mathcal{R}\left(\theta, \delta\left(B^H\right)\right) < \mathcal{R}\left(\theta, B^H\right) \qquad \text{for all } \theta \in M. \tag{4.2}$$

4.2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper. The fractional Brownian motion B^H has the following stochastic integral representation (see for instance, [1], [82])

$$B_t^{H,i} = \int_0^t K_H(t,s)dW_s^i, \quad i = 1,...,d; \quad t \in [0,T] \tag{4.3}$$

where $W = (W^1, ..., W^d)$ denotes the d-dimensional Brownian motion and the kernel $K_H(t,s)$ is equal to

$$c_H(t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2} - H)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - (\frac{s}{u})^{\frac{1}{2}-H}\right) du \quad \text{if } H \leq \frac{1}{2}$$
$$c_H(H - \frac{1}{2})\int_s^t (u-s)^{H-\frac{3}{2}}\left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{if } H > \frac{1}{2},$$

if $s < t$ and $K_H(t,s) = 0$ if $s \geq t$. Here c_H is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2-2H)}}$$

where Γ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [97].

The left fractional Riemann-Liouville integral of $f \in L^1((a,b))$ of order $\alpha > 0$ on (a,b) is given at almost all $x \in (a,b)$ by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)dy.$$

If $f \in I_{a^+}^{\alpha}(L^p(a,b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of f of order α defined by

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}dy\right)$$

for almost all $x \in (a,b)$.
For $H \in (0,1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s)f(s)ds$$

is a isomorphism from $L^2([0,1])$ onto $I_{0^+}^{H+\frac{1}{2}}\left(L^2([0,1])\right)$ and its inverse operator K_H^{-1} is given by

$$K_H^{-1} f = t^{H-\frac{1}{2}} D_{0^+}^{H-\frac{1}{2}} t^{\frac{1}{2}-H} f' \quad \text{for } H > 1/2, \tag{4.4}$$

$$K_H^{-1} f = t^{\frac{1}{2}-H} D_{0^+}^{\frac{1}{2}-H} t^{H-\frac{1}{2}} D_{0^+}^{2H} f \quad \text{for } H < 1/2. \tag{4.5}$$

Moreover, for $H < \frac{1}{2}$, if f is an absolutely continuous function then $K_H^{-1} f$ can be represented of the form (see [84])

$$K_H^{-1} f = t^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} t^{\frac{1}{2}-H} f'. \tag{4.6}$$

4.3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = \left(\theta^1, \ldots, \theta^d\right)$ belonging to M. An estimator $\xi = (\xi^1, \ldots, \xi^d)$ of $\theta = (\theta^1, \ldots, \theta^d)$ is called unbiased if, for every $t \in [0,T]$

$$E_\theta(\xi_t^i) = \theta_t^i, \qquad i = 1, \ldots, d$$

and it is called adapted if, for each $i = 1, \ldots, d$, ξ^i is adapted to $\left(\mathcal{F}_t^i\right)_{t\in[0,T]}$.
Since for any $i = 1, ..., d$, the function θ^i is deterministic and

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem (see Theorem 2 in [84]), yields that there exists a probability measure P_θ absolutely continuous with respect to P under which the process $\widetilde{B}^H := \left(\widetilde{B}_t^H; t \in [0,T]\right)$ defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0,T] \tag{4.7}$$

is a d-dimensional fBm with Hurst parameter H and mean zero. Moreover the Girsanov density of P_θ with respect to P is given by:

$$\frac{dP_\theta}{dP} = \exp\left[\sum_{i=1}^d \left(\int_0^T K_H^{-1}(\theta^i)(s)dW_s^i - \frac{1}{2}\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds\right)\right]$$

and

$$\widetilde{B}_t^H = \int_0^t K_H(t,s)d\widetilde{W}_s$$

where $\widetilde{W}$ is a d-dimensional Brownian motion under the probability P_θ and

$$\widetilde{W}_t^i = W_t^i - \int_0^t K_H^{-1}(\theta^i)(s)ds, \quad i = 1, ..., d; \quad t \in [0, T].$$

The equation (4.7) implies that B^H is an unbiased and adapted estimator of θ under probability P_θ. In addition, we obtain the Cramer-Rao type bound:

$$R(H, \hat{\theta}) := \mathcal{R}(\theta, B^H) = \int_0^T E_\theta \|\widetilde{B}_t^H\|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d.$$

The first main result of this section is given by the following proposition which asserts that $\widehat{\theta} = B^H$ is an efficient estimator of θ.

Theorem 10. *Assume that $H < \frac{1}{2}$. If ξ is an unbiased and adapted estimator of θ, then*

$$E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt \geq R(H, \hat{\theta}). \tag{4.8}$$

Proof: Since ξ is unbiased, then for every $\varphi \in M$ we have

$$E_\varphi(\xi_t^j) = E_\varphi(\varphi_t^j), \qquad j = 1, \ldots, d.$$

Let $\varphi = \theta + \varepsilon\psi$ with $\psi \in M$ and $\varepsilon \in \mathbb{R}$. Then for every $t \in [0, T]$ and $j \in \{1, \ldots, d\}$, we have

$$\begin{aligned} E_{\theta+\varepsilon\psi}(\xi_t^j) &= E_{\theta+\varepsilon\psi}(\theta_t^j + \varepsilon\psi_t^j) \\ &= E_{\theta+\varepsilon\psi}(\theta_t^j) + \varepsilon\psi_t^j. \end{aligned}$$

This implies that for every $j = 1, \ldots, d$

$$\begin{aligned} \psi_t^j &= \frac{d}{d\varepsilon}_{/\varepsilon=0} E_{\theta+\varepsilon\psi}(\xi_t^j - \theta_t^j) \\ &= E\left(\frac{d}{d\varepsilon}_{/\varepsilon=0} \exp\left[\sum_{i=1}^d \left(\int_0^t K_H^{-1}(\theta^i + \varepsilon\psi^i)(s)dW_s^i \right.\right.\right. \\ &\qquad \left.\left.\left. - \frac{1}{2}\int_0^t (K_H^{-1}(\theta^i + \varepsilon\psi^i)(s))^2)ds\right)\right](\xi_t^j - \theta_t^j)\right) \\ &= E_\theta\left(\sum_{i=1}^d \left[\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\theta^i)(s)ds\right] \right. \\ &\qquad \left. \times (\xi_t^j - \theta_t^j)\right) \\ &= E_\theta\left(\sum_{i=1}^d \left[\int_0^t K_H^{-1}(\psi^i)(s)d\widetilde{W}_s^i\right](\xi_t^j - \theta_t^j)\right) \\ &= E_\theta\left(\left[\int_0^t K_H^{-1}(\psi^j)(s)d\widetilde{W}_s^j\right](\xi_t^j - \theta_t^j)\right). \end{aligned}$$

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_\theta)$, we obtain that for every $t \in [0,T]$

$$\begin{aligned}\|\psi_t\|^2 = \sum_{j=1}^{d}(\psi_t^j)^2 &\leq \sum_{j=1}^{d} E_\theta\left((\xi_t^j - \theta_t^j)^2\right) E_\theta\left(\left[\int_0^t K_H^{-1}(\psi^j)(s)d\widetilde{W}_s^j\right]^2\right)\\ &= \sum_{j=1}^{d} E_\theta\left[\left((\xi_t^j - \theta_t^j)^2\right)\int_0^t (K_H^{-1}(\psi^j)(s))^2 ds\right].\end{aligned}$$

We take for each $j = 1, \ldots, d$, $\psi_t^j = t^{2H}$. Since $t \longrightarrow t^{2H}$ is absolutely continuous function, then by (4.6), a simple calculation shows that

$$\begin{aligned}K_H^{-1}(t^{2H}) &= 2Ht^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} t^{H-\frac{1}{2}}\\ &= \frac{2H\beta(\frac{1}{2}-H, H+\frac{1}{2})}{\Gamma(\frac{1}{2}-H)} t^{H-1/2}\\ &= 2H(\Gamma(\frac{1}{2}+H))t^{H-1/2}.\end{aligned}$$

It is known that

$$0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1,2]. \tag{4.9}$$

Combining the facts that $z\Gamma(z) = \Gamma(z+1)$, $z > 0$, $2H \leq (H+\frac{1}{2})^2$ and (4.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2}+H))^2 E_\theta\left(\|\xi_t - \theta_t\|^2\right) \leq E_\theta\left(\|\xi_t - \theta_t\|^2\right).$$

Hence, by an integration with respect to dt, we get

$$R(H,\hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.$$

Therefore (4.8) is satisfied.

Corollary 1. *The process $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ.*

Proof: We have for every $\psi \in M$

$$\frac{d}{d\varepsilon}_{/\varepsilon=0} \exp\left[\sum_{i=1}^{d}\int_0^t K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s)dW_s^i - \frac{1}{2}\int_0^t (K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s))^2)ds\right] = 0.$$

Hence

$$\sum_{i=1}^{d}\left(\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s)ds\right) = 0.$$

Which implies that for every $i = 1, ..., d$

$$E\left(\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s)ds\right)^2 = 0.$$

Then, for each $i = 1, ..., d$

$$W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s)ds, \qquad t \in [0,T].$$

Therefore by (4.3), we obtain that $B^H = \hat{\theta}$.

4.4 Superefficient James-Stein type estimators

The aim of this section is to construct superefficient estimators of θ which dominate, under the usual quadratic risk, the natural MLE estimator B^H. The class of estimators considered here are of the form

$$\delta(B^H)_t = B_t^H + g(B_t^H, t), \qquad t \in [0, T] \tag{4.10}$$

where $g : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ is a function. The problem turns to find sufficient conditions on g such that $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$ and the risk difference is negative, i.e.

$$\Delta\mathcal{R}(\theta) = \mathcal{R}\left(\theta, \delta(B^H)\right) - \mathcal{R}\left(\theta, B^H\right) < 0.$$

In the sequel we assume that the function g satisfies the following assumption:

$$(A)\begin{cases} E_\theta\left[\int_0^T ||g(B_t^H, t)||_d^2\, dt\right] < \infty, \\ \text{the partial derivatives } \partial_i g^i := \frac{\partial g^i}{\partial x^i},\ i = 1, \ldots, n \text{ of } g \text{ exist.} \end{cases}$$

Then $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$. Moreover

$$\begin{aligned} \Delta\mathcal{R}(\theta) &= E_\theta\left[\int_0^T ||B_t^H + g(B_t^H, t) - \theta_t||_d^2 - ||B_t^H - \theta_t||_d^2 dt\right] \\ &= E_\theta\left[\int_0^T ||g(B_t^H, t)||_d^2 + 2\sum_{i=1}^d \left(g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i)\right) dt\right]. \end{aligned}$$

In addition,

$$\begin{aligned} & E_\theta \int_0^T \sum_{i=1}^d \left(g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i)\right) dt \\ = & \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} g^i(x^1, \ldots, x^d, t)(x^i - \theta_t^i)\right. \\ & \qquad \left. \times\, e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \ldots dx^d\right) dt \\ = & \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t)\right. \\ & \qquad \left. \times\, e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \ldots dx^d\right) dt \\ = & \sum_{i=1}^d \int_0^T \left(t^{2H} E_\theta \partial_i g^i(B_t^H, t)\right) dt = E_\theta\left[\sum_{i=1}^d \int_0^T t^{2H} \partial_i g^i(B_t^H, t) dt\right]. \end{aligned}$$

Consequently, the risk difference equals

$$\Delta\mathcal{R}(\theta) = E_\theta\left[\int_0^T \left(||g(B_t^H, t)||^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t)\right) dt\right]. \tag{4.11}$$

We can now state the following theorem.

Theorem 11. *Let $g : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $\left(B_t^H + g(B_t^H, t)\right)_{t\in[0,T]}$ to dominate B^H under the usual quadratic risk which is equivalent that*

$$E_\theta\left[\int_0^T \left(||g(B_t^H,t)||^2 + 2t^{2H}\sum_{i=1}^d \partial_i g^i(B_t^H,t)\right)dt\right] < 0.$$

As an application, take g of the form

$$g(x,t) = at^{2H}\frac{r\left(\|x\|^2\right)}{\|x\|^2}x, \tag{4.12}$$

where a is a constant strictly positive and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded derivable function.
The next lemma give a sufficient condition for g in (4.12) to satisfies the assumption (A).

Lemma 5. *If $d \geq 3$ and $H < \frac{1}{2}$ then*

$$E\left[\int_0^T \frac{1}{\|B_t^H\|^2}dt\right] < \infty. \tag{4.13}$$

Proof: Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)\left((t,w); B_t^H(w) = 0\right) = 0$$

where $(dt \times P)$ is the product measure.
Using the change of variable and $d \geq 3$ we see that

$$E\int_0^T \frac{1}{\|B_t^H\|^2}dt = \int_0^T \frac{dt}{t^{2H}}\int_{\mathbb{R}^d}\frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi}\|y\|^2}dy \leq C\int_0^T \frac{1}{t^{2H}}dt,$$

where C is a constant depending only on d. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

Theorem 12. *Assume that $d \geq 3$. If the function r, the constant a and the parameter H satisfy:*

i) $0 \leq r(.) \leq 1$

ii) $r(\cdot)$ *is differentiable and increasing*

iii) $0 < a \leq 2(d-2)$ *and* $H < 1/2$,

then the estimator

$$\delta(B^H) = B_t^H - at^{2H}\frac{r\left(\|B_t^H\|^2\right)}{\|B_t^H\|^2}B_t^H, \qquad t \in [0,T].$$

dominates B^H.

Proof: It suffices to prove that $\Delta\mathcal{R}(\theta) < 0$. From (4.11) and the hypothesis *i)* and *ii)* we can write

$$\begin{aligned}\Delta\mathcal{R}(\theta) &= aE_\theta\left[\int_0^T t^{4H}\left(\frac{ar^2(\|B_t^H\|^2)}{\|B_t^H\|^2} - 2(d-2)\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right.\right.\\ &\qquad\left.\left. - 4r'(\|B_t^H\|^2)\right)dt\right]\\ &\le a\left[a-2(d-2)\right]E_\theta\left[a\int_0^T t^{4H}\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right].\end{aligned}$$

Combining this fact with the assumption *iii)* yields that the risk difference is negative. Which proves the desired result.

For $r = 1$, we obtain a James-Stein type estimator:

Corollary 2. *Let $d \ge 3$, $0 < H < \frac{1}{2}$ and $0 < a \le 2(d-2)$. Then the estimator*

$$\left(1 - \frac{at^{2H}}{\|B_t^H\|^2}\right)B_t^H, \qquad t \in [0,T]$$

dominates B^H.

Part III

CONTRIBUTIONS TO THE STUDY OF LÉVY PROCESSES

Chapter 5

Lévy processes and Itô-Skorohod integrals

We study Skorohod integral processes on Lévy spaces and we prove an equivalence between this class of processes and the class of Itô-Skorohod process (in the sense of [109]).

5.1 Introduction

We study in this work anticipating integrals with respect to a Lévy process. The anticipating integral on the Wiener space, known in general as the Skorohod integral (and sometimes as the Hitsuda integral) constitutes an extension of the standard Itô integral to non-adapted integrands. It is nothing else than the classical Itô integral if the integrand is adapted. The Skorohod integral has been extended to the Poisson process and next it has been defined with respect to a normal martingale (see [29]) due to the Fock space structure generates by such processes. Recently, an anticipating calculus of Malliavin-type has been defined on Lévy spaces again by using some multiple stochastic integral with respect to a Lévy process which have been in essence defined in the old paper by K. Itô (see [55]). We refer to [30], [31] or [106] for Malliavin calculus on Lévy spaces and possible applications to mathematical finance.

The purpose of this paper is to understand the relation between anticipating Skorohod integral processes and Itô-Skorohod integral process (in the sense of [109] or [93]) in the Lévy case. We recall that the results in [109] and [93] show that on Wiener and Poisson spaces the class of Skorohod integral process with regular integrals coincides with the class of some Itô-Skorohod integrals that have similarities to the classical Itô integrals for semimartingales. The fact that the driven processes have independent increments plays an crucial role. Therefore, it is expected to obtain the same type of results for Lévy processes. We prove here such an equivalent between Skorohod and Itô-Skorohod integrals by using a recent Malliavin calculus for Lévy processes.

Section 2 contains some preliminaries on Lévy processes and Malliavin-Skorohod calculus for them. In Section 3 we prove a generalized Clark-Ocone formula that we will use in Section 4 to prove the correspondence between Skorohod and Itô-Skorohod integrals and to develop an

Itô-type calculus for anticipating integrals on Lévy spaces.

5.2 Preliminaries

In this section we introduce the basic properties of Malliavin calculus for Lévy processes used in this paper. For more details the reader is referred to [106].

In this paper we deal with a càdlàg Lévy process $X = (X_t)_{0\leq t\leq 1}$ defined on a certain complete probability space $\big(\Omega, (F_t^X)_{0\leq t\leq 1}, P\big)$, with the time horizon $T = [0,1]$, and equipped with its generating triplet (γ, σ^2, ν) where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and $\nu(dz)$ is the Lévy measure on $\mathbb{R}$ which, we recall, is such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} 1 \wedge x^2 \nu(dx) < \infty$$

Throughout the paper, we suppose that $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$, and we use the notation and terminologies of [3], [106]. By N we will denote the jump measure of X:

$$N(E) = \#\{t : (t, \Delta X_t) \in E\},$$

for $E \in B(T \times \mathbb{R}_0)$, where $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $\Delta X_t = X_t - X_{t-}$, and # denotes the cardinal. We will note $\widetilde{N}$ the compensated jump measure:

$$\widetilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx).$$

The process X admits a Lévy-Itô representation

$$X_t = \gamma t + \sigma W_t + \int\int_{(0,t]\times\{|x|>1\}} xN(ds, dx) + \lim_{\varepsilon\downarrow 0} \int\int_{(0,t]\times\{\varepsilon<|x|\leq 1\}} x\widetilde{N}(ds, dx)$$

where W is a standard Brownian motion.
Define

$$\mu(E) = \sigma^2 \int_{E(0)} ds + \int\int_{\{E-E(0)\times\{0\}\}} x^2 ds\nu(dx).$$

Itô [55] proved that X can be extended to a martingale-valued measure M of type $(2, \mu)$ on $(T \times R, B(T \times \mathbb{R}))$: For any $E \in B(T \times\ \mathbb{R})$ with $\mu(E) < \infty$

$$M(E) = \sigma \int_{E(0)} dW_s + \lim_{n\to\infty} \int\int_{\{(s,x)\in E: \frac{1}{n}<|x|<n\}} x\widetilde{N}(ds, dx).$$

where $E(0) = \{s \in T : (s, 0) \in E\}$.
Furthermore, M is a centered independent random measure such that

$$E\left(M(E_1)M(E_2)\right) = \mu(E_1 \cap E_2)$$

for any $E_1, E_2 \in B(T \times \mathbb{R})$ with $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$.

Using the random measure M one can construct multiple stochastic integrals driven by a Lévy process as an isometry between $L^2(\Omega)$ and the space $L^2\left((T\times R)^n, B((T\times R)^n), \mu^{\otimes n}\right)$. Indeed, one can use the same steps as on the Wiener space: first, consider a simple function f of the form

$$f = 1_{E_1\times\ldots\times E_n}$$

where $E_1,\ldots,E_n \in B(T\times\mathbb{R})$ are pathwise disjoint and $\mu(E_i) < \infty$ for every i. For such a function, define $I_n(f) = M(E_1)\ldots M(E_n)$ and then the operator I_n can be extended by linearity and continuity to an isometry between $L^2(\Omega)$ and the space $L^2\left((T\times\mathbb{R})^n, B((T\times\mathbb{R})^n), \mu^{\otimes n}\right)$.

An interesting fact is that, as in the Brownian and Poissonian cases, M enjoys the chaotic representation property (see [106]), i.e. every $F\in L^2(\Omega, F^X, P) = L^2(\Omega)$, can be written as an orthogonal sum of multiple stochastic integrals

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$$

where the convergence holds $L^2(\Omega)$ and $f_n \in L^2_s\left((T\times\mathbb{R})^n, B((T\times\mathbb{R})^n), \mu^{\otimes n}\right)$ (the last space is the space of symmetric and square integrable functions on $(T\times\mathbb{R})^n$ with respect to $\mu^{\otimes n}$).

At this point we can introduce a Malliavin calculus with respect to the Lévy process X by using this Fock space-type structure. If

$$\sum_{n=0}^{\infty} nn!\|f_n\|_n^2 < \infty \tag{5.1}$$

(here $\|f_n\|_n$ denotes the norm in the space $L^2\left((T\times R)^n, B((T\times R)^n), \mu^{\otimes n}\right)$) then the Malliavin derivative of F is introduced as an annihilation operator (see for example [85])

$$D_zF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(z,.)), \quad z\in T\times\mathbb{R}.$$

The domain of the operator D is denoted by $D^{1,2}$. It is exactly the set of random variables verifying (5.1). We denote by $D^{k,2},\ k\geq 1$ the domain of the kth iterated derivative $D^{(k)}$, which is a Hilbert space with respect the scalar product

$$\langle F, G\rangle = E(FG) + \sum_{j=1}^{k} E\int_{(T\times R)^j} D_z^{(j)}FD_z^{(j)}G\mu(dz).$$

We introduce now the Skorohod integral with respect to X as a creation operator. Let $u\in H = L^2\left(T\times R\times\Omega, B(T\times\mathbb{R})\otimes F_T^X, \mu\otimes P\right)$, then, for every $z\in T\times\mathbb{R}$, $u(z)$ admit the following representation

$$u(z) = \sum_{n=0}^{\infty} I_n(f_n(z,.)).$$

Here we have $f_n\in L^2((T\times\mathbb{R})^{n+1}, \mu^{\otimes n+1})$ and f_n is symmetric in the last n variables. If

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|_{n+1}^2 < \infty \tag{5.2}$$

($\tilde{f}_n$ represents the symmetrization of f_n in all its $n+1$ variables) then the Skorohod integral $\delta(u)$ of u with respect to X is introduced by

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

The domain of δ is the set of processes satisfying (5.2) and we have the duality relationship

$$E(F\delta(u)) = E\int\int_{T\times R} D_z F u(z)\mu(dz), \quad F \in D^{1,2}.$$

We will use the notation

$$\delta(u) = \int_0^1\int_{\mathbb{R}} u_z \delta M(dz) = \int_0^1\int_{\mathbb{R}} u_{s,x}\delta M(ds,dx).$$

Remark 5. *It has been proved in [106] that if the integrand is predictable then the Skorohod integral coincides with the standard semi-martingale integral introduced in [3].*

For $k \geq 1$, we denote by $L^{k,2}$ the set $L^2((T\times\mathbb{R}; D^{k,2}), \mu)$. In particular one can prove that $L^{1,2}$ is given by the set of u in the above chaotic form such that (5.2) satisfied.
We also have $L^{k,2} \subset Dom\delta$ for $k \geq 1$ and and for every $u, v \in L^{1,2}$

$$E(\delta(u)\delta(v)) = E\int\int_{T\times\mathbb{R}} u(z)v(z)\mu(dz) + E\int\int_{(T\times\mathbb{R})^2} D_z u(z')D_{z'}v(z)\mu(dz)\mu(dz').$$

In particular

$$E(\delta(u))^2 = E\int\int_{T\times\mathbb{R}} u(z)^2\mu(dz) + E\int\int_{(T\times\mathbb{R})^2} D_z u(z')D_{z'}u(z)\mu(dz)\mu(dz').$$

The commutativity relationship between the derivative operator and Skorohod integral is given by: let $u \in L^{1,2}$ such that $D_z u \in Dom(\delta)$, then $\delta(u) \in D^{1,2}$ and

$$D_z\delta(u) = u(z) + \delta(D_z(u)), \quad z \in T\times R.$$

5.3 Generalized Clark-Ocone formula

We start this section by proving some properties of the multiple integrals $I_n(f)$ and how it behaves if it is conditioned by a σ-algebra. If $A \in B(T)$ we will denote by F_A^X the σ-algebra generated by the increments of the process X on the set A

$$F_A^X = \sigma(X_t - X_s : s,t \in A).$$

Proposition 6. *Let* $f \in L_s^2((T \times \mathbb{R})^n, \mu^{\otimes^n})$ *and* $A \in B(T)$. *Then*

$$E\left(I_n(f)/F_A^X\right) = I_n(f 1_{(A\times\mathbb{R})}^{\otimes^n}).$$

Proof: By density and linearity argument, it is enough to consider $f = 1_{E_1\times...\times E_n}$, where $E_1, ..., E_n$ are pairwise disjoint set of $B(T \times R)$ and $\mu(E_i) < \infty$ for every $i = 1, ..., n$. In this case we have

$$\begin{aligned} E\left(I_n(f)/F_A^X\right) &= E\left(M(E_1)....M(E_n)/F_A^X\right) \\ &= E\left(\prod_{i=1}^{n} \left(M(E_i \cap (A\times\mathbb{R})) + M(E_i \cap (A^c \times \mathbb{R}))\right)/F_A^X\right) \\ &= \prod_{i=1}^{n} M(E_i \cap (A\times\mathbb{R})) = I_n(f 1_{(A\times\mathbb{R})}^{\otimes^n}). \end{aligned}$$

And an immediate consequence, we have

Corollary 3. *Suppose that* $F \in D^{1,2}$ *and* $A \in B(T)$. *Then the conditional expectation* $E(F/F_A^X)$ *belongs to* $D^{1,2}$ *and for every* $z \in T \times \mathbb{R}$

$$D_z E\left(F/F_A^X\right) = E\left(D_z F/F_A^X\right) 1_{A\times\mathbb{R}}(z).$$

Proof: let F have the form $\sum_{n\geq 0} I_n(f_n)$ with $f_n \in L_s^2\left((T\times\mathbb{R})^n, B((T\times\mathbb{R})^n), \mu^{\otimes n}\right)$. Then, by Proposition 8,

$$D_z E\left(F/F_A^X\right) = D_z\left(\sum_{n\geq 0} I_n\left(f_n 1_{A\times\mathbb{R}}^{\otimes n}\right)\right)$$

$$= \sum_{n\geq 1} n I_{n-1}\left(f_n(\cdot, z) 1_{A\times\mathbb{R}}^{\otimes(n-1)}\right) 1_{A\times R}(z)$$

and it remains to observe that $E\left(D_z F/F_A^X\right) = \sum_{n\geq 1} n I_{n-1}\left(f_n(\cdot, z) 1_{A\times\mathbb{R}}^{\otimes(n-1)}\right)$.

At this point we can state the following version of Clark-Ocone formula on Lévy space. This extends a results in [106].

Proposition 7 (Generalized Clark-Ocone formula)**.** *Let* F *be a random variable in* $D^{1,2}$. *Then, for every* $0 \leq s < t \leq 1$, *we have*

$$F = E\left(F/F_{(s,t]^c}^X\right) + \delta(h_{s,t}(\cdot))$$

where for $(r, x) \in T \times R$ *we denoted by* $h_{s,t}(r, x) = E\left(D_{r,x}F/F_{(r,t]^c}\right) 1_{(s,t]^c}(r)$.*Moreover*

$$F = E\left(F/F_{(s,t]^c}^X\right) + \int\int_{(s,t]\times\mathbb{R}} {}^{(p,t)}(D_z F)\ dM_z$$

$$= E\left(F/F^X_{(s,t]^c}\right) + \sigma \int_s^t {}^{(p,t)}(D_{r,0}F)dW_r + \int\int_{(s,t]\times\mathbb{R}_0} {}^{(p,t)}(D_{r,x}F)\ \widetilde{N}(dr,dx)$$

where ${}^{(p,t)}(DF)$ *is the predictable projection of* DF *with respect to the filtration* $\left(F^X_{(r,t]^c}\right)_{r\le t}$.

Proof: Let F have the form $\sum_{n=0}^{\infty} I_n(f_n)$, where $f_n \in L^2_s\left(([0,1]\times\mathbb{R})^n, \mu^{\otimes n}\right)$. Firstly, we prove that

$$F = E\left(F/F^X_{(s,t]^c}\right) + \delta\left(h_{s,t}(\cdot)\right).$$

Indeed, for any $s < t \le 1$ we have

$$E(D_{r,x}F/F^X_{(r,t]^c})1_{(s,t]\times\mathbb{R}}(r,x) = \sum_{n=1}^{\infty} nI_{n-1}\left[f_n((r,x),.)1^{\otimes n-1}_{(r,t]^c\times\mathbb{R}}(.)\right]1_{(s,t]\times\mathbb{R}}(r,x).$$

Hence, using that $x\in\mathbb{R}$ and thus the symmetrization with respect to the variable x has no effect, we obtain

$$\delta\left(h_{s,t}\right) = \sum_{n=1}^{\infty} nI_n\left[f_n((t_1,x_1),...,(t_n,x_n))\widetilde{1^{\otimes n-1}_{(t_1,t]^c}(t_2,...,t_n)1_{(s,t]}(t_1)}\right]$$

Since

$$\widetilde{1^{\otimes n-1}_{(t_1,t]^c}(t_2,...,t_n)1_{(s,t]}(t_1)}$$

$$= \frac{1}{n!}\sum_{i=1}^{n}\sum_{\sigma(1)=i,\sigma\in S_n}^{n} 1^{\otimes n-1}_{(t_i,t]^c}(t_{\sigma(2)},...,t_{\sigma(n)})1_{(s,t]}(t_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 1^{\otimes n-1}_{(t_i,t]^c}(t_1,...,\widehat{t_i},...,t_n)1_{(s,t]}(t_i)$$

$$= \frac{1}{n}\left(1 - 1^{\otimes n}_{(s,t]^c}(t_1,...,t_n)\right).$$

Then

$$\delta\left(h_{s,t}\right) = F - E\left(F/F^X_{(s,t]^c}\right).$$

The second equality in the statement follows from [106] (see also [93]) where the equivalence of the two representations has been proven.

5.4 Itô-Skorohod integral calculus

As a consequence of the above results, we will show in this part that every Skorohod integral process of the form

$$Y_t := \delta(u_{.}1_{[0,t]\times\mathbb{R}}(\cdot)), \quad t\in T$$

can be written as an Itô-Skorohod integral in the sense of [109] (this integral has similarities with the standard stochastic integral). In this way we will extend in the Lévy case the results of [109] concerning the Wiener case and of [93] concerning the Poisson case. The key point of our construction is the fact that the driving process has independent increments.

Proposition 8. *Assume that $u \in L^{k,2}$ with $k \geq 3$. Then there exist an unique process $v \in L^{k-2,2}$ such that, for every $t \in T$,*

$$Y_t := \delta(u_{.}1_{[0,t]\times\mathbb{R}}(\cdot)) = \int\int_{(0,t]\times\mathbb{R}} {}^{(p,t)}(v_{s,x})\ M(ds,dx).$$

Moreover $v_{s,x} = D_{s,x}Y_s \quad \mu \otimes P$ a.e. on $T \times R \times \Omega$.

Proof: Applying above generalized Clark-Ocone formula, we have

$$Y_t = E\left(Y_t/F^X_{(0,t]^c}\right) + \int\int_{(0,t]\times\mathbb{R}} {}^{(p,t)}(D_zY_t)\ dM_z.$$

The process Y satisfies (see Lemma 3.2.1 in [81])

$$E\left(Y_t - Y_s/F^X_{(s,t]^c}\right) = 0$$

for every $s < t$. Indeed, take a $F^X_{(s,t]^c}$-measurable random variable F in $D^{1,2}$. According to the duality relationship and Corollary 1, we have

$$E(F(Y_t - Y_s)) = E[F\delta(u_{.}1_{(s,t]\times\mathbb{R}}(.))] = E\left\langle D_{.}F, u_{.}1_{(s,t]\times\mathbb{R}}(.)\right\rangle_{L^2(T\times\mathbb{R},\ \mu)} = 0.$$

Therefore, we obtain

$$E\left(Y_t/F^X_{(0,t]^c}\right) = E\left(Y_t - Y_0/F^X_{(0,t]^c}\right) = 0,$$

and, using corollary 3, we have

$$\begin{aligned}
\delta\left[{}^{(p,t)}(D_{s,x}Y_t)\,1_{[0,t]\times R}(s,x)\right] &= \delta\left[E\left(D_{s,x}Y_t/F^X_{(s,t]^c}\right)1_{[0,t]\times R}(s,x)\right]\\
&= \delta\left[D_{s,x}E\left(Y_t/F^X_{(s,t]^c}\right)1_{[0,t]\times\mathbb{R}}(s,x)\right]\\
&= \delta\left[D_{s,x}E\left(Y_s/F^X_{(s,t]^c}\right)1_{[0,t]\times\mathbb{R}}(s,x)\right]\\
&= \delta\left[E\left(D_{s,x}Y_s/F^X_{(s,t]^c}\right)1_{[0,t]\times\mathbb{R}}(s,x)\right]\\
&= \delta\left[{}^{(p,t)}(D_{s,x}Y_s)\,1_{[0,t]\times\mathbb{R}}(s,x)\right]
\end{aligned}$$

We thus have

$$Y_t = \delta\left[{}^{(p,t)}(D_{s,x}Y_s)\,1_{[0,t]\times\mathbb{R}}(s,x)\right] = \int\int_{(0,t]\times\mathbb{R}} {}^{(p,t)}(D_{s,x}Y_s)\ M(ds,dx).$$

Taking $v_{s,x} = D_{s,x}X_s$. To obtain the desired Itô-Skorohod representation it is sufficient to prove that $v \in L^{k-2,2}$. By using the property of commutativity between D and δ and the inequalities for the norms of anticipating integrals we have

$$\|v\|^2_{1,2} \leq \|u\|^2_{1,2} + \|\delta(D_{s,x}u_{.}1_{[0,s]\times R}(.))\|^2_{1,2}$$

$$\leq \|u\|^2_{1,2} + E\int\int_{T\times R} \left(\delta(D_{s,x}u_{.}1_{[0,s]\times R}(.)\right)^2 \mu(ds,dx)$$

$$+E\int\int_{(T\times R)^2}\left(D_{r,y}\delta(D_{s,x}u_{\cdot}1_{[0,s]\times R}(\cdot))\right)^2\mu(ds,dx)\mu(dr,dy)$$

$$\leq \|u\|_{1,2}^2+3E\int_{T\times\mathbb{R}}\int_{T\times R}\left(D_{z_2}u_{z_1}\right)^2\mu(z_1)\mu(z_2)$$

$$+3E\int_{T\times\mathbb{R}}\int_{T\times\mathbb{R}}\int_{T\times R}\left(D_{z_3}D_{z_2}u_{z_1}\right)^2\mu(z_1)\mu(z_2)\mu(z_3)$$

$$+2E\int_{T\times\mathbb{R}}\int_{T\times\mathbb{R}}\int_{T\times\mathbb{R}}\int_{T\times\mathbb{R}}\left(D_{z_4}D_{z_3}D_{z_2}u_{z_1}\right)^2\mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\leq 4\|u\|_{3,2}^2.$$

The same manner, we found that

$$\|v\|_{k-2,2}^2\leq C(k)\|u\|_{k,2}^2,$$

where $C(k)$ is a positive constant depending of k. To conclude our proof we have to show the uniqueness of these processes. We assume that there exist v and v' in $L^{k-2,2}$ such that

$$Y_t=\int\int_{(0,t]\times R}{}^{(p,t)}(v_{s,x})\ M(ds,dx)=\int\int_{(0,t]\times\mathbb{R}}{}^{(p,t)}(v'_{s,x})\ M(ds,dx).$$

Using again the property of commutativity, we have

$$E\left(w_{s,x}/F^X_{[s,t]^c}\right)1_{[0,t]\times\mathbb{R}}(s,x)+\int\int_{(s,t]\times\mathbb{R}}E\left(D_{s,x}w_{r,x}/F^X_{[r,t]^c}\right)\ M(dr,dx)=0,$$

where $w_{s,x}=v_{s,x}-v'_{s,x}$. Conditioning by $F^X_{[s,t]^c}$, we obtain

$$E\left(w_{s,x}/F^X_{[s,t]^c}\right)1_{[0,t]\times\mathbb{R}}(s,x)=0,\quad s\leq t,x\in\mathbb{R}.$$

By letting t goes to s we get that $w_{s,x}=0$ in $L^2(\Omega)$ for every $(s,x)\in\ T\times\ \mathbb{R}$. We can thus conclude that $v=v'$ in $L^{k-2,2}$.

Using the correspondence between Skorohod and Itô-Skorohod integrals proved above, we can derive an Itô formula for anticipating integrals on Lévy space. As far as we know, this is the only Itô formula proved for these class of processes.

Proposition 9 (Itô's formula)**.** *Let v be a process belonging to $L^2(T\times\mathbb{R}\times\Omega,\mu\otimes P)$. Let us consider the following stochastic process*

$$Y_t=\int\int_{(0,t]\times\mathbb{R}}E\left(v_{s,x}/F^X_{[s,t]^c}\right)M(ds,dx)$$

and let f be a C^2 real function. Then

$$\begin{aligned}f(Y_t)\ &=\ f(0)+\int\int_{(0,t]\times\mathbb{R}}f'(Y_t^{s^-})^{(p,t)}(D_{s,x}Y_s)\ M(ds,dx)\\&+\frac{1}{2}\int\int_{(0,t]\times\mathbb{R}}f''(Y_t^{s^-})(^{(p,t)}(D_{s,0}Y_s))^2\ ds\\&+\sum_{0<s\leq t}\left(f(Y_t^s)-f(Y_t^{s^-})-f'(Y_t^{s^-})(Y_t^s-Y_t^{s^-})\right)\end{aligned}$$

where $Y_t^s:=\int\int_{(0,s]\times\mathbb{R}}E\left(v_{s,x}/F^X_{[s,t]^c}\right)M(ds,dx)$ and $Y_t^{s^-}\ =\ \lim_{r\to s^-}Y_t^r$ for all $0<s\leq t$.

Proof: Fix $t \in (0,T]$. We define $Z_s = Y_t^s$ if $s \leq t$ and $Z_s = Y_t$ if $s > t$. Also let $(G_s)_{s\geq 0}$ be a filtration given as follows $G_s = F^X_{[s,t]^c}$ if $s \leq t$ and $G_s = F^X_1$, if $s > t$.
It is easy to see that $(Z_s)_{s\geq 0}$ is a square integrable càdlàg martingale with respect to $(G_s)_{s\geq 0}$.

Applying Itô's formula (see [99], Theorem 32, p. 71) we obtain for every $s > 0$

$$\begin{aligned} f(Z_s) &= f(0) + \int_{(0,t]} f'(Z_{s^-})dZ_s + \frac{1}{2}\int_{(0,t]} f''(Z_{s^-})d[Z,Z]^c_s \\ &\quad + \sum_{0<s\leq t} \left(f(Z_s) - f(Z_{s^-}) - f'(Z_{s^-})(Z_s - Z_{s^-})\right) \end{aligned}$$

where $[Z,Z]^c$ is the continuous part of quadratic variation process $[Z,Z]$ of Z. It is well known that $[N,N]^c_s = 0$. From proposition 4 of [106], we see that for every $s \leq t$

$$[Z,Z]^c_s = [Y,Y]^c_s = \sigma^2 \int_{(0,s]} E\left(v_{r,0}/F^X_{[r,t]^c}\right)^2 dr.$$

Thus, and in particular for $s = t$

$$\begin{aligned} f(Z_t) = f(Y_t) &= f(0) + \int\int_{(0,t]\times R} f'(Y_t^{s^-}) E\left(v_{r,x}/F^X_{[r,t]^c}\right) M(ds,dx) \\ &\quad + \frac{1}{2}\int\int_{(0,t]\times R} f''(Y_t^{s^-})\left[E\left(v_{r,0}/F^X_{[r,t]^c}\right)\right]^2 ds \\ &\quad + \sum_{0<s\leq t}\left(f(Y_t^s) - f(Y_t^{s^-}) - f'(Y_t^{s^-})(Y_t^s - Y_t^{s^-})\right), \end{aligned}$$

so that the desired conclusion follows.

Another consequence of Proposition 3 is the following Burkholder inequality which gives a bound for the L^p norm of the anticipating integral.

Proposition 10 (Burkholder's Inequality)**.** *Let Y be a process of Itô-Skorohod form as above and $2 \leq p < \infty$. Then there exist a universal constant $C(p)$ such that*

$$E|Y_t|^p \leq C(p) E\left(\sigma^2 \int_{(0,t]} E\left(v_{r,0}/F^X_{[r,t]^c}\right)^2 dr + \int\int_{(0,t]\times R_0} x^2 E\left(v_{r,x}/F^X_{[r,t]^c}\right)^2 N(dr,dx)\right)^{p/2}.$$

Proof: The proof of this proposition is straightforward from Theorem 54, P. 174 in [99] and the approximation procedure used along the paper.

Chapter 6

How rich is the class of processes which are infinitely divisible with respect to time?

We give a link between stochastic processes which are infinitely divisible with respect to time (IDT) and Lévy processes. We investigate the connection between the selfsimilarity and the strict stability for IDT processes. We also consider a subordination of a Lévy process by an increasing IDT process. We introduce a notion of multiparameter IDT stochastic processes, extending the one studied by Mansuy [72]. The main example of this kind of processes is the Lévy sheet.

6.1 Introduction

An $\mathbb{R}^d-$valued stochastic process $X = (X_t, t \geq 0)$ is said to be IDT if, for every $n \in \mathbb{N}$, we have

$$(X_{nt}, t \geq 0) \overset{d}{=} (X_t^{(1)} + ... + X_t^{(n)}, t \geq 0), \tag{6.1}$$

where $(X^{(1)}, t \geq 0), ..., (X^{(n)}, t \geq 0)$ are independent copies of X and $\overset{d}{=}$ denotes equality in all finite-dimensional distributions. The notion of IDT processes has been introduced by Mansuy [72] as a generalization of Lévy processes. Various properties of IDT processes have been already investigated in [72], related for instance to their temporal self-decomposability and the characterization of IDT Gaussian processes. Regarded as a contribution to this expending topic, it is the purpose of this paper to extend some results on Lévy processes studied in [7], [35] and [73] to the case of IDT processes. In particular, we shall prove that IDT processes are more tractable than Lévy processes, since they could be obtained by combining the selfsimilarity and strict stability. Such a result is not true in general for Lévy processes. Moreover, we will prove that a necessary and sufficient condition for an IDT process to be a Lévy process is the hypothesis of independence increments. Actually, this last condition can be circumvented when dealing with IDT processes. So it turns out that the class of IDT processes is very rich.

The paper is organized as follows. Section 2 contains some preliminaries on stable processes and selfsimilar processes. Section 3 establishes, for IDT processes, the connection between the selfsimilar (semi-selfsimilar, resp.) processes and the strictly stable (strictly semi-stable, resp.) processes. Namely, strictly stable (strictly semi-stable, resp.) IDT process is a simple example of selfsimilar (semi-selfsimilar, resp.) process. As a byproduct, we consider the so-called Lamperti transformation for strictly semi-stable IDT processes to give a generalized semi-stable Ornstein-Uhlenbeck process (see Definition 3.4).

Time-changed Lévy processes where the chronometers are more general than subordinators arise now in many fields of application, see for instance [7] and the reference therein. We shall prove (Theorem 3.6) the inheritance of IDT under time change when base processes are Lévy processes.

In section 4 we shall introduce a notion of multiparameter IDT processes and we give several examples of this kind of processes, one of them is the Lévy sheet. Contrary to the one-parameter case, we will prove that multiparameter Lévy processes are not IDT in our sense. As in the one-parameter case [72], we characterize the multiparameter IDT Gaussian processes. Moreover, we define multiparameter temporal selfdecomposable processes similar to those introduced by Barndorff-Nielsen, Maejima and Sato [7] and we prove that multiparameter IDT processes are temporal selfdecomposable.

6.2 Preliminaries

In this section we recall some definitions that are needed in the sequel. For more details the reader is referred to Sato [105].

An $\mathbb{R}^d-$valued random variable X is called degenerate if it is a constant almost surely. An $\mathbb{R}^d-$valued process $(X_t, t \geq 0)$ is called trivial if X_t is degenerate for every t.

Let $0 < \alpha \leq 2$. An infinitely divisible probability measure μ on $\mathbb{R}^d$ is called $\alpha-$stable if, for any $a > 0$, there is $\gamma_a \in \mathbb{R}^d$ such that

$$\hat{\mu}(\theta)^a := \left(\int_{\mathbb{R}^d} e^{i<\theta,z>} \mu(dz) \right)^a = \hat{\mu}(a^{1/\alpha}\theta) e^{i<\theta,\gamma_a>}, \ \forall\ \theta \in \mathbb{R}^d. \tag{6.2}$$

It is called strictly $\alpha-$stable if, for any $a > 0$,

$$\hat{\mu}(\theta)^a = \hat{\mu}(a^{1/\alpha}\theta), \ \forall\ \theta \in \mathbb{R}^d. \tag{6.3}$$

It is called $\alpha-$semi-stable if, for some $a > 0$ with $a \neq 1$, there is $\gamma_a \in \mathbb{R}^d$ satisfying (6.2). It is called strictly $\alpha-$semi-stable if, there is some $a > 0$ with $a \neq 1$ satisfying (6.3).

Let $(X_t, t \geq 0)$ be a Lévy (IDT, resp.) process on $\mathbb{R}^d$. It is called a α-stable, strictly α-stable, semi α-stable, or strictly α-semi-stable Lévy (IDT, resp.) process if every finite-dimensional distribution of X is, respectively, α-stable, strictly α-stable, semi α-stable, or strictly α-semi-stable.

Let $H > 0$. A stochastic process $(X_t, t \geq 0)$ on $\mathbb{R}^d$ is called H-selfsimilar if, for any $a > 0$,

$$(X_{at}, t \geq 0) \stackrel{d}{=} (a^H X_t, t \geq 0). \tag{6.4}$$

It is called wide-sense H-selfsimilar if, for any $a > 0$, there is a function $c(t)$ from $\mathbb{R}^+$ to $\mathbb{R}^d$ such that

$$(X_{at}, t \geq 0) \stackrel{d}{=} (a^H X_t + c(t), t \geq 0). \tag{6.5}$$

It is called H-semi-selfsimilar if there is some $a > 0$ with $a \neq 1$ satisfying (6.4). It is called wide-sense H-semi-selfsimilar if, for some $a > 0$ with $a \neq 1$, there is a function $c(t)$ satisfying (6.5).

6.3 Stable IDT processes

The goal of this section is to generalize some properties of Lévy process to the case of IDT process. We first establish rather a link between IDT process and Lévy process than between IDT process and selfsimilar process.

Theorem 13. *If $X = (X_t, t \geq 0)$ is an IDT, stochastically continuous process with independent increments, then X is a Lévy process.*

Proof: It suffices to prove that X has stationary increments. Using the IDT property (6.1), we obtain

$$Ee^{i\theta X_k} = \left(Ee^{i\theta X_1}\right)^k, \quad \text{for any } k \in \mathbb{N}. \tag{6.6}$$

In a similar way (6.6) can be obtained when k is a rational time. It follows now from the stochastic continuity of X that

$$Ee^{i\theta X_t} = \left(Ee^{i\theta X_1}\right)^t, \quad \text{for any } t \in \mathbb{R}^+.$$

Therefore, for any $0 \leq s < t$, we have

$$Ee^{i\theta X_{t-s}} \quad = \quad \left(Ee^{i\theta X_1}\right)^{t-s} = \frac{Ee^{i\theta X_t}}{Ee^{i\theta X_s}} = Ee^{i\theta (X_t - X_s)},$$

where the last equality follows from the independence of increments. Since for IDT processes $X_0 = 0$ almost surely, then X has stationary increments, which completes the proof.

Remark 6. *If $(X_t, t \geq 0)$ is a centered Gaussian process satisfying the assumptions of the previous proposition, then X is Brownian motion up to a multiplicative constant, that is has the covariance function $c(s,t) = E(X_t X_s) = (s \wedge t)\ c(1,1)$. In particular, for Gaussian processes, one can replace the assumption of independence of increments by stationarity of increments.*

Indeed, let $s < t$, since X is an IDT centered Gaussian process, then it is $1/2$-selfsimilar (see [72]), hence

$$E\exp\left(i(X_t - X_s)\right) = \exp\left(-\frac{1}{2}\left[(t+s)EX_1^2 - 2E(X_tX_s)\right]\right). \tag{6.7}$$

On the other hand, we have

$$E\exp\left(i(X_t - X_s)\right) = \left(E\exp(iX_1)\right)^{t-s} = \exp\left(-\frac{1}{2}(t-s)E(X_1)^2\right). \tag{6.8}$$

It follows from (6.7) and (6.8) that $E(X_tX_s) = sE(X_1)^2$ for $s < t$.

Proposition 11. *Let $0 < \alpha \leq 2$. A nontrivial, strictly α-stable and $(1/\alpha)$-selfsimilar process $(X_t, t \geq 0)$ is an IDT process.*

Proof: Since X is strictly α-stable, we have

$$\left(n^{(1/\alpha)}X_t,\ t \geq 0\right) \stackrel{d}{=} \left(\sum_{i=1}^{n} X_t^{(i)},\ t \geq 0\right), \quad \forall\, n \in \mathbb{N}^*,$$

where $X^{(1)}, ..., X^{(n)}$ are independent copies of X. On the other hand, it follows from the self-similarity of X that

$$(X_{nt}, t \geq 0) \stackrel{d}{=} \left(n^{(1/\alpha)}X_t,\ t \geq 0\right),$$

which implies X is an IDT process.

6.3.1 Strictly stable IDT processes

In the case of a Lévy process $(X_t, t \geq 0)$, Theorem 1.4.2 in [35] prove that $\mathcal{L}(X_1)$ is stable if and only if $(X_t, t \geq 0)$ is selfsimilar. We can generalize this result as follows.

Theorem 14. *Let $(X_t, t \geq 0)$ be a nontrivial, stochastically continuous and IDT process. Then $(X_t, t \geq 0)$ is strictly α-stable if and only if it is $(\frac{1}{\alpha})$-selfsimilar.*

Proof: First, assume that $(X_t, t \geq 0)$ is $(\frac{1}{\alpha})$-selfsimilar. Using the IDT property, we obtain

$$\left(n^{(\frac{1}{\alpha})}X_t, t \geq 0\right) \stackrel{d}{=} (X_{nt}, t \geq 0) \stackrel{d}{=} \left(\sum_{i=1}^{n} X_t^{(i)}, t \geq 0\right), \quad \text{for any } n \in \mathbb{N}^*,$$

where $X^{(1)}, ..., X^{(n)}$ are independent copies of X. Thus X is strictly α-stable.
Conversely, suppose that $(X_t, t \geq 0)$ is strictly α-stable. Since X is an IDT process, we have

$$(X_{nt}, t \geq 0) \stackrel{d}{=} (\sum_{i=1}^{n} X_t^{(i)}, t \geq 0) \stackrel{d}{=} \left(n^{\frac{1}{\alpha}}X_t, t \geq 0\right), \quad \text{for any } n \in \mathbb{N}^*,$$

and also

$$\left(X_{\frac{t}{n}}, t \geq 0\right) \stackrel{d}{=} \left((\frac{1}{n})^{\frac{1}{\alpha}}X_t, t \geq 0\right).$$

Hence, for any $m \in \mathbb{N},\ n \in \mathbb{N}^*$, we have that

$$\left(X_{(\frac{m}{n}t)}, t \geq 0\right) \stackrel{d}{=} \left((m/n)^{\frac{1}{\alpha}} X_t, t \geq 0\right).$$

Combining this last fact with the stochastic continuity of X, we obtain that X is $(\frac{1}{\alpha})$-selfsimilar. The proof is now complete.

Next, we will give an example of an IDT process which is not a Lévy process and satisfies the above theorem.

Example 2. *Let S_α be a strictly α-stable random variable. The process X defined by*

$$X_t = t^{1/\alpha} S_\alpha,\ t \geq 0,$$

is an $(1/\alpha)$-selfsimilar and IDT process.

Corollary 4. *If $\alpha = 2$ or $0 < \alpha < 1$, then all $\frac{1}{\alpha}$-selfsimilar IDT process with stationary increments is an α-stable Lévy processes.*

The proof of this corollary is straightforward from Theorem 7.5.4 in [104] and Theorem 14 above.

Corollary 5 (Sub-stable processes)**.** *Let $0 < \alpha < 2$, $\alpha < \beta \leq 2$ and $(Y_t, t \geq 0)$ be a symmetric β-stable IDT process. Let ξ be a (α/β)-stable positive random variable independent of Y. The process $(X_t, t \geq 0)$, defined by*

$$X_t = \xi^{\frac{1}{\beta}} Y_t,$$

is $(\frac{1}{\beta})$-selfsimilar, symmetric α-stable.

Proof: The $1/\beta$-self similarity follows from Theorem 14 while the symmetric α-stability can be proved by using classical arguments on sub-stable processes (see Example 3.6.3 in [35]). Details are left to the reader.

6.3.2 Strictly semi-stable IDT processes

The following result gives the connection between semi-selfsimilarity and strict semi-stability for IDT processes. The case of Lévy processes appears in [[105], Theorem 4.1].

Theorem 15. *Let $(X_t, t \geq 0)$ be an $\mathbb{R}^d$-valued IDT, stochastically continuous process. Then*
1) $(X_t, t \geq 0)$ is semi-stable if it is wide-sense semi-selfsimilar.
2) $(X_t, t \geq 0)$ is semi-selfsimilar if and only if it is strictly semi-stable.

Proof: 1) Suppose X is wide-sense H-semi-selfsimilar. Then for some $a \in (0,1) \cup (1,\infty)$, there exists a nonrandom function $c : [0,\infty) \to \mathbb{R}^d$, such that

$$(X_{at}, t \geq 0) \stackrel{d}{=} (a^H X_t + c(t), t \geq 0).$$

Therefore, and by using IDT property, for all $t_1, t_2, ..., t_m \in \mathbb{R}_+$ and all $(\theta_1, ...\theta_m) \in \mathbb{R}^{d\times m}$, we have

$$\begin{aligned}\left(Ee^{i\sum_{k=1}^m \langle \theta_k, X_{t_k}\rangle}\right)^a &= \left(Ee^{i\sum_{k=1}^m \langle \theta_k, X_{at_k}\rangle}\right)\\ &= \left(Ee^{i\sum_{k=1}^m \langle \theta_k, a^H X_{t_k}\rangle}\right)\left(e^{i\sum_{k=1}^m \langle \theta_k, c(t_k)\rangle}\right).\end{aligned}$$

It remains to show that $(X_{t_1}, ..., X_{t_m})$ is infinitely divisible, which follows from the IDT property. Thus $(X_t, t \geq 0)$ is $1/H$-semi-stable.
2) Assume that $(X_t, t \geq 0)$ is strictly α-semi-stable. Then, for some $a \in (0,1) \cup (1, \infty)$, we have

$$\begin{aligned}\left(Ee^{i\langle \theta, a^{1/\alpha}(X_{t_1}, ..., X_{t_m})\rangle}\right) &= \left(Ee^{i\langle \theta, (X_{t_1}, ..., X_{t_m})\rangle}\right)^a\\ &= \left(Ee^{i\langle \theta, (X_{at_1}, ..., X_{at_m})\rangle}\right),\end{aligned}$$

where the last equality follows from the IDT property. Hence $(X_t, t \geq 0)$ is $1/\alpha$-semi-selfsimilar. The converse can be proved in a similar way.

Is a wide-sense H-selfsimilar in fact H-selfsimilar, if it is H-semi-selfsimilar? The answer is known in the case of a stable Lévy process, see [73]. We can also answer this question in the case of a stable IDT process.

Proposition 12. *Let $(X_t, t \geq 0)$ be an $\mathbb{R}^d$-valued, nontrivial and stochastically continuous α-stable IDT process. If it is strictly α-semi-stable, then it is strictly α-stable.*

Proof: Assume that $(X_t, t \geq 0)$ is strictly α-semi-stable. Then, for some $a > 1$,

$$(X_{at}, t \geq 0) \stackrel{d}{=} (a^{1/\alpha} X_t, t \geq 0). \tag{6.9}$$

Fix $t_1, ..., t_m \in \mathbb{R}_+$ and $b_1, ..., b_m \in \mathbb{R}$. Then there exists a finite measure Γ on the unit sphere S of $\mathbb{R}^{d\times m}$, a vector μ in $\mathbb{R}^{d\times m}$ and a symmetric nonnegative-definite matrix A, such that the characteristic function of $X := (X_{t_1}, ..., X_{t_m})$ has the following form

$$\begin{aligned}&Ee^{i<\theta,X>}\\ &= \begin{cases} e^{\left\{-\int_S |<\theta,s>|^\alpha (1 - i sign(<\theta,s>)\tan(\pi\alpha/2))\Gamma(ds) + i<\theta,\mu>\right\}} & \text{if } \alpha \neq 1,2 \\ e^{\left\{-\int_S |<\theta,s>|\left(1 + i\frac{2}{\pi} sign(<\theta,s>)\ln|<\theta,s>|\right)\Gamma(ds) + i<\theta,\mu>\right\}} & \text{if } \alpha = 1 \\ e^{\left\{-\frac{1}{2}<A\theta,\theta> + i<\theta,\mu>\right\}} & \text{if } \alpha = 2. \end{cases}\end{aligned}$$

The pair (Γ, μ) is unique, and

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Using the IDT property and (6.9), we have

$$\left(Ee^{i<\theta,X>}\right)^a = Ee^{i<\theta,(X_{at_1},...,X_{at_m})>} = Ee^{i<a^{\frac{1}{\alpha}}\theta,X>}, \quad \forall\, \theta \in \mathbb{R}^{d\times m}. \tag{6.10}$$

First, if $\alpha \neq 1$, then according to (6.10) we have $a\mu = a^{\frac{1}{\alpha}}\mu$. This implies that $\mu = 0$. Thus, X is strictly α-stable.
Let us now assume that $\alpha = 1$. Then

$$\int_S <\theta, s> \ln| <\theta, s> |\Gamma(ds) = \int_S <\theta, s> \ln| <a\theta, s> |\Gamma(ds).$$

Consequently

$$\int_S \ln(a) <\theta, s> \Gamma(ds) = 0, \ \forall \theta \in \mathbb{R}^d.$$

This means that $\int_S s_k \Gamma(ds) = 0$ for $k = 1, 2, ..., d$. Which is exactly the condition for the strictly $1-$stability of X. Then every finite-dimensional distribution of $(X_t, t \geq 0)$ is strictly α-stable. The proof is completed.

Applying the Lamperti transformation to semi-stable IDT processes, we derive a new class of periodically stationary processes. Recall that a stochastic process $(Y_t, t \in \mathbb{R})$ is said to be periodically stationary with period p (> 0) if

$$(Y_{t+p}, t \in \mathbb{R}) \stackrel{d}{=} (Y_t, t \in \mathbb{R}).$$

Definition 1. *Let $0 < \alpha \leq 2$ and $(X_t, t \geq 0)$ be a strictly α-semi-stable IDT and stochastically continuous process. We define a periodically stationary process $(Y_t, t \in \mathbb{R})$ by*

$$Y_t = e^{-t/\alpha} X_{e^t}.$$

We call this process a generalized α-semi-stable Ornstein-Uhlenbeck process.

Proposition 13. *A generalized α-semi-stable Ornstein-Uhlenbeck process $(Y_t, t \in \mathbb{R})$ is strictly α-semi-stable.*

Proof: From Theorem 15, point 2) we obtain

$$(X_{at}, t \geq 0) \stackrel{d}{=} (a^{1/\alpha} X_t, t \geq 0), \quad \text{for some} \ \ a \in (0,1) \cup (1, \infty). \tag{6.11}$$

Combining the IDT property and (6.11), we have that for any $\theta = (\theta_1, ..., \theta_m)$, $\theta_k \in \mathbb{R}^d$, $k = 1, ..., m$ and $(t_1, ..., t_m) \in \mathbb{R}^m$,

$$\begin{aligned} \left(Ee^{i\left\langle \theta,(Y_{t_1},...,Y_{t_m})\right\rangle}\right)^a &= \left(Ee^{i\left\langle \theta,(e^{-t_1/\alpha}X_{ae^{t_1}},...,e^{-t_m/\alpha}X_{ae^{t_m}})\right\rangle}\right) \\ &= \left(Ee^{i\left\langle \theta,(a^{1/\alpha}e^{-t_1/\alpha}X_{e^{t_1}},...,a^{1/\alpha}e^{-t_m/\alpha}X_{e^{t_m}})\right\rangle}\right) \\ &= \left(Ee^{i\left\langle a^{1/\alpha}\theta,(Y_{t_1},...,Y_{t_m})\right\rangle}\right). \end{aligned}$$

Thus $(Y_{t_1}, ..., Y_{t_m})$ is strictly α-semi-stable for any $(t_1, ..., t_m) \in \mathbb{R}^m$. The proof is completed.

6.3.3 Subordination through an IDT process

Subordination is a transformation of a stochastic process to a new stochastic process through random time change by an increasing Lévy process (subordinator) independent of the original process. The aim of this paragraph is to investigate the case where the chronometer is an increasing IDT process .

Definition 2. *A real-valued stochastic process* $\xi = \{\xi_t, t \geq 0\}$ *with* $\xi_0 = 0$ *a.s. is called a chronometer if it is increasing and stochastically continuous.*

The following result is inheritance of IDT under time change when initial processes are Lévy processes.

Theorem 16. *Let* X *be a Lévy process on* $\mathbb{R}^d$ *and* ξ *is an IDT chronometer such that* X *and* ξ *are independent. Then* $(Z_t := X_{\xi_t} : t \geq 0)$ *is an IDT process.*

Proof: Let $\xi^{(j)}$, $j = 1, ..., n$ be independent copies of ξ. Since X is independent of ξ, then for every $n \geq 1$, $\theta = (\theta_1, ..., \theta_m) \in (\mathbb{R}^d)^m$, we have

$$J(n, \theta) := E \exp\{\sum_{k=1}^{m} i \left\langle \theta_k, X_{\xi_{nt_k}} \right\rangle\} \quad = \quad E \left[\left(E \exp\{\sum_{k=1}^{m} i \left\langle \theta_k, X_{s_k} \right\rangle\} \right)_{s_k = \xi_{nt_k},\ k=1,...,m} \right].$$

By using the IDT property, we obtain

$$J(n, \theta) = E \left[\left(E \exp\{\sum_{k=1}^{m} i \left\langle \theta_k, X_{s_k} \right\rangle\} \right)_{s_k = \sum_{j=1}^{n} \xi^{(j)}_{t_k},\ k=1,...,m} \right].$$

According to the change of variables $c_k = \theta_k + ... + \theta_n$ and $t_0 = 0$, and the independence of increments of X, we have

$$\begin{aligned} J(n, \theta) &= E \left[\left(E \exp\{\sum_{k=1}^{m} i \left\langle c_k, X_{s_k} - X_{s_{k-1}} \right\rangle\} \right)_{s_k = \sum_{j=1}^{n} \xi^{j}_{t_k},\ k=1,...,m} \right] \\ &= E \left[\left(\prod_{k=1}^{m} E \exp\{i \left\langle c_k, X_{s_k} - X_{s_{k-1}} \right\rangle\} \right)_{s_k = \sum_{j=1}^{n} \xi^{(j)}_{t_k},\ k=1,...,m} \right]. \end{aligned}$$

Now, it follows from the stationarity of the increments of X and the independence of the $\xi^{(j)}$, $j = 1, ..., n$, that

$$\begin{aligned} J(n, \theta) &= E \left[\left(\prod_{k=1}^{m} \prod_{j=1}^{n} E \exp\{i \left\langle c_k, X_{r_k^j} \right\rangle\} \right)_{r_k^j = \xi^{(j)}_{t_k} - \xi^{(j)}_{t_{k-1}},\ k=1,...,m,\ j=1,...,n} \right] \\ &= E \left[\left(\prod_{j=1}^{n} E \exp\{\sum_{k=1}^{m} i \left\langle \theta_k, X_{r_k^j} \right\rangle\} \right)_{r_k^j = \xi^{(j)}_{t_k},\ k=1,...,m,\ j=1,...,n} \right] \end{aligned}$$

$$= \left(E\left[\left(E\exp\{\sum_{k=1}^{m} i\left\langle \theta_k, X_{r_k}\right\rangle\}\right)_{r_k=\xi_{t_k},\ k=1,\dots,m}\right]\right)^n$$
$$= \left(E\exp\{\sum_{k=1}^{m} i\left\langle \theta_k, X_{\xi_{t_k}}\right\rangle\}\right)^n .$$

This completes the proof.

Similarly, one can prove the following result.

Proposition 14. *Let $\left\{X_s : s \in \mathbb{R}_+^N\right\}$ be an $\mathbb{R}_+^N$-parameter Lévy process on $\mathbb{R}^d$ and let $\{\xi_t : t \geq 0\}$ be a N-dimensional subordinator in the sense of being a N-dimensional IDT process $\{\xi_t\} = \left\{\left(\xi_t^1, ..., \xi_t^N\right)^\top\right\}$ that is increasing in each coordinate with the superscript $\top$ denoting the transpose, and $\{\xi_t\}$ independent of $\left(X(s) : s \in \mathbb{R}_+^N\right)$. Define the subordinated process by composition as follows*

$$Y_t = X_{\xi_t}, \quad t \geq 0.$$

Then $(Y_t : t \geq 0)$ is an IDT process on $\mathbb{R}^d$.

6.4 Multiparameter IDT processes

In this section we introduce the notion of multiparameter infinitely divisible with respect to time (IDT) process. A typical example of this processes is the Lévy sheet.

Definition 3. *An $\mathbb{R}^d$-valued stochastic process $(X_t, t \in \mathbb{R}_+^N)$ is said to be IDT if for any $n = (n_1, ..., n_N) \in (\mathbb{N}^*)^N$,*

$$\left(X_{(n.t)}, t \in \mathbb{R}_+^N\right) \overset{d}{=} \left(\sum_{i=1}^{\prod_{k=1}^N n_k} X_t^{(i)},\ t \in \mathbb{R}_+^N\right);$$

where $X^{(1)}, ..., X^{(\prod_{k=1}^N n_k)}$ are independent copies of X and $(n.t) := (n_1 t_1, ..., n_N t_N)$.

In the following we give some examples of multiparameter IDT processes.

Example 3. *1) Let ξ be a strictly α-stable random variable. The process defined by*

$$\left(X_t = (t_1^{1/\alpha} t_2^{1/\alpha} .... t_N^{1/\alpha})\xi,\ t \in \mathbb{R}_+^N\right),$$

is an IDT process.

2) If X is an IDT process and μ a measure on $\mathbb{R}_+^N$ such that

$$X_t^{(\mu)} = \int_{\mathbb{R}_+^N} X_{(s.t)}\ \mu(ds),\ t \in \mathbb{R}_+^N$$

is well defined, then $X^{(\mu)}$ is an IDT process.

3) Let $(X_t, t \geq 0)$ be an IDT process. Then the multiparameter process defined by

$$Y_t = X_{t_1 t_2 \ldots t_N}, \qquad \text{for any } t = (t_1, ..., t_N) \in \mathbb{R}_+^N,$$

is IDT.

In order to show that any Lévy sheet process is IDT, we give firstly the definition of such a process.

Definition 4. *Let $(X_t, t \in \mathbb{R}_+^2)$ be a family of random variables on $\mathbb{R}^d$. We write X_{s_1,s_2} instead of X_s when $s = (s_1, s_2)^\top$. For $s = (s_1, s_2)^\top$ and $u = (u_1, u_2)^\top$ in $\mathbb{R}_+^2$ with $s_1 \leq u_1$ and $s_2 \leq u_2$, call $B = (s_1, u_1] \times (s_2, u_2]$ a rectangle in $\mathbb{R}_+^2$ and set*

$$X(B) = X_{u_1,u_2} - X_{s_1,u_2} - X_{u_1,s_2} + X_{s_1,s_2}.$$

If $B_1, ..., B_n$ are disjoint rectangles in $\mathbb{R}_+^2$ and $B = \cup_{j=1}^n B_j$, then set $X(B) = \sum_{j=1}^n X(B_j)$. The stochastic process $(X_t, t \in \mathbb{R}_+^2)$ is called a Lévy sheet if

(a) If $n \geq 2$ and $B_1, ..., B_n$ are disjoint rectangles, then $X(B_1), ..., X(B_n)$ are independent.

(b) If B is a rectangle and $s \in \mathbb{R}_+^2$, then $X(B) \stackrel{d}{=} X(B + s)$.

(c) $X_{s_1,0} = X_{0,s_2} = 0$ a.s. for $s_1, s_2 \in \mathbb{R}_+^2$.

(d) $X_t \to X_s$ in probability as $|t - s| \to 0$ in $\mathbb{R}_+^2$.

Proposition 15. *Let $(X_t, t \in \mathbb{R}_+^2)$ be a Lévy sheet process on $\mathbb{R}^d$. Then it is IDT.*

Proof: Let $n, m \in \mathbb{N}$, $\theta = (\theta_1, ..., \theta_p) \in \mathbb{R}^{d \times p}$, $0 = s_0 \leq s_1 < ... < s_p$ and $0 = t_0, t_1, ..., t_p \in \mathbb{R}_+$, let σ be a permutation such that $t_{\sigma(1)} \leq ... \leq t_{\sigma(p)}$ and $\sigma(0) = 0$. We consider disjoint rectangles $\left(B_k^l = (ns_{k-1}, ns_k] \times (mt_{\sigma(l-1)}, mt_{\sigma(l)}]\right)$, $k = 1, ..., p, l = 1, ..., p$. Using Lévy sheet properties (see [94]), there exist a matrix $\left(c_k^l\right)_{0 \leq k,l \leq p}$, $c_k^l \in \mathbb{R}^d$ such that

$$\begin{aligned}
Ee^{i\sum_{j=1}^p \left\langle \theta_j, X_{(ns_j, mt_j)} \right\rangle} &= Ee^{i\sum_{j=1}^p \langle \theta_j, X((0,ns_j] \times (0,mt_j]) \rangle} = Ee^{i\sum_{0 \leq k,l \leq p} \left\langle c_k^l, X(B_k^l) \right\rangle}. \\
&= \prod_{0 \leq k,l \leq p} Ee^{i\left\langle c_k^l, X(B_k^l) \right\rangle} \\
&= \prod_{0 \leq k,l \leq p} \left(Ee^{i\left\langle c_k^l, X_{1,1} \right\rangle}\right)^{\lambda(B_k^l)} = I, \qquad \text{say,}
\end{aligned}$$

where λ is the Lebesgue measure. Since $\lambda(B_k^l) = n \times m\lambda\left((s_{k-1}, s_k] \times (t_{\sigma(l)}, t_{\sigma(l-1)}]\right)$, then we have

$$\begin{aligned}
I &= \left(\prod_{0 \leq k,l \leq p} \left(Ee^{i\left\langle c_k^l, X_{1,1} \right\rangle}\right)^{\lambda\left((s_{k-1}, s_k] \times (t_{\sigma(l)}, t_{\sigma(l-1)}]\right)}\right)^{n \times m} \\
&= \left(\prod_{0 \leq k,l \leq p} \left(Ee^{i\left\langle c_k^l, X\left((s_{k-1}, s_k] \times (t_{\sigma(l)}, t_{\sigma(l-1)}]\right) \right\rangle}\right)\right)^{n \times m} \\
&= \left(Ee^{i\sum_{j=1}^p \langle \theta_j, X((0,s_j] \times (0,t_j]) \rangle}\right)^{n \times m}
\end{aligned}$$

This completes the proof.

Remark 7. *If $(X_t, t \in \mathbb{R}_+^N)$ is an $\mathbb{R}^d$-valued stochastically continuous IDT process, then*

$$X_t = 0 \text{ a.s for any } t \in \mathbb{R}_+^N, \text{ with } \inf_{i=1,\dots,N} t_i = 0.$$

Indeed, for any $n \geq 1$, $u \in \mathbb{R}^d$ and $t = (t_1, ..., t_{j-1}, 0, t_{j+1}, ...t_N)$, by IDT property we have

$$E \exp\left(i\langle u, X_t\rangle\right) = E \exp\left(i\langle u, X_{(t_1,\dots,t_{j-1},n\times 0,t_{j+1},\dots,t_N)}\rangle\right) = \left[E \exp\left(i\langle u, X_t\rangle\right)\right]^n.$$

Moreover the characteristic function of X_t is non vanishing (because the laws of this variable is infinitely divisible), then

$$Ee^{iuX_t} = 1, \text{ for all } u \in \mathbb{R}^d.$$

Thus, $\mathcal{L}(X_t) = \delta_0$, where δ_0 is the distribution concentrated at 0 and $\mathcal{L}(X)$ denotes the law of X.

According to Mansuy [72], Proposition 1.1, any one-parameter Lévy process is IDT. The following result shows that such a result does not hold in the multiparameter case. We refer to Pedersen and Sato [[94], Definition 2.1] for the definition and properties of $\mathbb{R}_+^N$-parameter Lévy processes in law.

Proposition 16. *Let $N \geq 2$. If $(X_t, t \in \mathbb{R}_+^N)$ is an $\mathbb{R}_+^N$-parameter Lévy processes in law satisfying the IDT property, then $X_t = 0$ a.s, $\forall t \in \mathbb{R}_+^N$.*

Proof: Let X be such a process. Then, by Remark 7, we have

$$Ee^{i\langle u,X_t\rangle} = \prod_{k=1}^{N} Ee^{i\left\langle u,X_{(0,\dots,0,t_k,0,\dots,0)}\right\rangle} = 1, \text{ for any } t \in \mathbb{R}_+^N,\ u \in \mathbb{R}^d.$$

This proves that $\mathcal{L}(X_t) = \delta_0,\ \forall t \in \mathbb{R}_+^N$.

In the following we show that for any stochastically continuous IDT process X there exists a Lévy sheet that has the same one-dimensional marginals than X.

Proposition 17. *Let $\left(X_t, t \in \mathbb{R}_+^2\right)$ be a stochastically continuous IDT process. Then there exists a Lévy sheet process $(Z_t, t \in \mathbb{R}_+^2)$ such that*

$$X_t \overset{d}{=} Z_t, \qquad \text{for any } t \in \mathbb{R}_+^2.$$

Proof: First we note that the laws of finite dimensional marginals of an multiparameter IDT process X are infinitely divisible. In particular, the law of $X_{1,1}$ is infinitely divisible. Then there exists a Lévy sheet $(Z_t, t \in \mathbb{R}_+^2)$ with $X_{1,1} \overset{d}{=} Z_{1,1}$. It follows that if $Z_t, t \in \mathbb{R}_+^2$ is a Lévy sheet then $\mathcal{L}(X_t)$ is infinitely divisible and

$$E\left[e^{i\left\langle z,X_{t_1,t_2}\right\rangle}\right] = \left(E\left[e^{i\langle z,X_{1,1}\rangle}\right]\right)^{t_1t_2} \qquad \text{for any } t_1, t_2 \in \mathbb{R}_+.$$

Hence

$$X_t \overset{d}{=} Z_t, \qquad \text{for any } t \in \mathbb{R}_+^2.$$

Definition 5. *An stochastic process* $\left(Y_t, t \in \mathbb{R}^N\right)$ *is said to be strictly stationary if, for every* $a \in \mathbb{R}^N$

$$\left(Y_{a+t}, t \in \mathbb{R}^N\right) \stackrel{d}{=} \left(Y_t, t \in \mathbb{R}^N\right).$$

For the characterization of IDT Gaussian multiparameter processes, we will need the following:

Lemma 6. *Let* $\left(Y_t, t \in \mathbb{R}^N\right)$ *be a strictly stationary process, and fix* $H = (h_1, ..., h_N) \in (\mathbb{R}_+^*)^N$. *Define* $\left(X_t, t \in \mathbb{R}_+^N\right)$ *by*

$$X_t = t_1^{h_1} ... t_N^{h_N} Y_{(log(t_1),...,log(t_N))}, \; t \in (\mathbb{R}_+)^N; \; X_0 = 0.$$

Then $\left(X_t, t \in \mathbb{R}_+^N\right)$ *is* H*-selfsimilar in the following sense*

$$\left(X_{(a.t)}, t \in (\mathbb{R}_+)^N\right) \stackrel{d}{=} \left(a_1^{h_1} ... a_N^{h_N} X_t, t \in (\mathbb{R}_+)^N\right) \text{ for any } a \in (\mathbb{R}_+^*)^N.$$

Conversely, if $\left(X_t, t \in (\mathbb{R}_+)^N\right)$ *is* H*-selfsimilar process, then its Lamperti transform*

$$\left(Y_t := e^{\{-\sum_{k=1}^N h_k t_k\}} X_{(e^{t_1},...,e^{t_N})}, \; t \in \mathbb{R}^N\right),$$

is strictly stationary.

Proof: Assume that $\left(Y_t, t \in \mathbb{R}^N\right)$ is strictly stationary. Then, for any $a = (a_1, ..., a_N) \in (\mathbb{R}_+^*)^N$, we have

$$\begin{aligned}
\left(X_{(a.t)}, t \in \mathbb{R}_+^N\right) &= \left((a_1 t_1)^{h_1} ... (a_N t_N)^{h_N} Y_{(log(a_1)+log(t_1),...,log(a_N)+log(t_N))}, t \in \mathbb{R}_+^N\right) \\
&\stackrel{d}{=} \left((a_1 t_1)^{h_1} ... (a_N t_N)^{h_N} Y_{(log(t_1),...,log(t_N))}, t \in \mathbb{R}_+^N\right) \\
&= \left(a_1^{h_1} ... a_N^{h_N} X_t, t \in \mathbb{R}_+^N\right).
\end{aligned}$$

Hence, $\left(X_t, t \in \mathbb{R}_+^N\right)$ is H-selfsimilar. Conversely, since $\left(X_t, t \in \mathbb{R}^N\right)$ is H-selfsimilar, for any $b = (b_1, ..., b_N) \in {\mathbb{R}_+}^N$ we have

$$\begin{aligned}
\left(Y_{(t+b)}, t \in \mathbb{R}^N\right) &= \left(e^{\{-\sum_{k=1}^N h_k (t_k + b_k)\}} X_{(e^{t_1+b_1},...,e^{t_N+b_N})}, \; t \in \mathbb{R}^N\right) \\
&\stackrel{d}{=} \left(e^{\{-\sum_{k=1}^N h_k t_k\}} X_{(e^{t_1},...,e^{t_N})}, \; t \in \mathbb{R}^N\right) = \left(Y_t, t \in \mathbb{R}^N\right).
\end{aligned}$$

Thus, $\left(Y_t, t \in \mathbb{R}^N\right)$ is strictly stationary.

Proposition 18. *Let* $(X_t, t \in \mathbb{R}_+^N)$ *be a stochastically continuous, and centered Gaussian process. Then the following properties are equivalent:*
i) $(X_t, t \in \mathbb{R}_+^N)$ *is an IDT process.*
ii) The covariance function $c(s,t) := E(X_s X_t)$, $(s,t) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$, *satisfies*

$$c(\alpha.s, \alpha.t) = \alpha_1 ... \alpha_N c(s,t), \text{ for any } \; \alpha \in (R_+^*)^N.$$

iii) The process $(X_t, t \in \mathbb{R}_+^N)$ *is* $(\frac{1}{2}, ..., \frac{1}{2})$*-selfsimilar.*
iv) $\left(Y_y := e^{\{-\sum_{k=1}^N h_k y_k\}} X_{(e^{y_1},...,e^{y_N})}, \; y \in \mathbb{R}^N\right)$ *is strictly stationary.*

Proof: $i) \Leftrightarrow ii)$. We have that $(X_t, t \in \mathbb{R}_+^N)$ is IDT if and only if, for any $n = (n_1, ..., n_N) \in \mathbb{N}^N,\ s, t \in \mathbb{R}_+^N$

$$c(n.s, n.t) = n_1...n_N c(s,t),$$

and also, for any $q = (q_1, ..., q_N) \in \mathbb{Q}_+^N,\ s, t \in \mathbb{R}_+^N$

$$c(q.s, q.t) = q_1...q_N c(s,t).$$

Moreover, since X is stochastically continuous, its covariance function is continuous. Hence, using the density of $\mathbb{Q}_+$ in $\mathbb{R}_+$, we obtain the result.
$ii) \Leftrightarrow iii)$. Since a centered Gaussian process is characterized by its covariance function, we obtain the result easily.
$iii) \Leftrightarrow iv)$. It is a direct application of Lemma 6 for $H = (1/2, ..., 1/2)$.

Example 4. *Let $(B_t,\ t \in \mathbb{R}_+^N)$ be a Brownian sheet, i.e. a centered, real-valued Gaussian random field with covariance function $E\left(B(t)B(s)\right) = \prod_{i=1}^N s_i \wedge t_i$. Since its covariance function satisfies the point ii) of Proposition 18, then $(B_t, t \in \mathbb{R}_+^N)$ is an IDT process.*

We will define the multiparameter temporally selfdecomposable processes which extend the one introduced in [7] and relate this notion with the IDT processes.

Definition 6. *An $\mathbb{R}^d$-valued stochastic process $X = (X_t, t \in \mathbb{R}_+^N)$ is temporally selfdecomposable if, for every $c \in (0,1)^N$, there exist two independent processes $X^{(c)} = (X_t^{(c)}, t \in \mathbb{R}_+^N)$, and $U^{(c)} = (U_t^{(c)}, t \in \mathbb{R}_+^N)$ on $\mathbb{R}^d$, such that*

$$X \stackrel{d}{=} X^{(c)} + U^{(c)},$$

where $X^{(c)} \stackrel{d}{=} (X_{(c.t)}, t \in \mathbb{R}_+^N)$ and U^c is called the c-residual of X.
For every $m \geq 2$, we say that X is temporally selfdecomposable of order m if, it is temporally selfdecomposable and if any $c \in (0,1)^N$, the c-residual process of X is temporally selfdecomposable of order $(m-1)$. When X is temporally selfdecomposable of order m for all m, we call it infinitely temporally selfdecomposable.

Proposition 19. *An $\mathbb{R}^d$-valued stochastically continuous IDT process is infinitely temporally selfdecomposable.*

Proof: By stochastic continuity and IDT property, we have, for any $(t_1, ..., t_m) \in (\mathbb{R}_+^N)^m$, $\xi \in \mathbb{R}^m$ and $c = (c_1, ..., c_N) \in (0,1)^N$

$$\begin{aligned} Ee^{\sum_{j=1}^m i\langle \xi_j, X_{t_j}\rangle} &= \left(Ee^{\sum_{j=1}^m i\left\langle \xi_j, X_{(c_1 t_j^1, ..., c_N t_j^N)}\right\rangle}\right)^{1/\prod_{k=1}^N c_k} \\ &= \left(Ee^{\sum_{j=1}^m i\langle \xi_j, X_{c.t_j}\rangle}\right)\left(Ee^{\sum_{j=1}^m i\left\langle \xi_j, X_{\left(c_1\left(\frac{1}{\prod_{k=1}^N c_k}-1\right)t_j^1, ..., c_N\left(\frac{1}{\prod_{k=1}^N c_k}-1\right)t_j^N\right)}\right\rangle}\right). \end{aligned}$$

Therefore $(X_t, t \in \mathbb{R}_+^N)$ is temporally selfdecomposable and

$$(X_t, t \in \mathbb{R}_+^N) \stackrel{d}{=} (X_{c.t} + U_t^c, t \in \mathbb{R}_+^N),$$

where U^c is independent of $(X_{c.t}, t \in \mathbb{R}_+^N)$ and

$$(U_t^c, t \in \mathbb{R}_+^N) \stackrel{d}{=} (X_{(c_1(\frac{1}{\prod_{k=1}^N c_k}-1)t^1,\ldots,c_N(\frac{1}{\prod_{k=1}^N c_k}-1)t^N)}, t \in \mathbb{R}_+^N). \tag{6.12}$$

It follows from (6.12) that U^c is stochastically continuous and IDT. The same steps as above applied to U^c proves that U^c is temporally selfdecomposable and its residuel process is stochastically continuous and IDT. Using the same arguments, we conclude that X is infinitely temporally selfdecomposable.

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