Detecting abrupt change on the Hurst parameter of a multi-scale fractional Brownian motion with applications

26 septembre 2001

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In some applications, for instance finance or biomecanic, it appears that it could be relevant to modelize data with a generalization of a fractional Brownian motion for which the Hurst parameter H is depending on the frequency. In this contribution, we describe the multiscale fractional Brownian motions which present a parameter H as a piece-wise constant function of the frequency. We describe the main properties of these processes and propose a statistical method based on wavelet analysis to detect changes on the parameter.

Keywords: Detection of change; Estimation of the Hurst parameter; Fractional Brownian motion; Wavelet analysis

1 Introduction

Fractional Brownian motions (f.B.m.) were popularized by Mandelbrot and Van Ness (1968) who suggest the study of their properties as a typical example of non-Markovian process. F.B.m. are Gaussian processes with stationary increments. These processes are self-similar and their increments are short or long memory processes and both these properties are controlled by the same parameter : the Hurst index H. But f.B.m. appears as an ideal mathematical model, in some applications real data lead to the modelization by locally self-similar processes with a time-varying Hurst index $h(t)$, see Cohen (2000) and the references therein. When the Hurst index is time-varying, the increments of the process are no more stationary.

In this paper, we consider Gaussian processes with stationary increments with a Hurst index varying with the frequencies. To our knowledge, these kinds of processes have been introduced (implicitly) by Collins & de Luca (1993) in a statistical study of the position of the center of pressure in upright position, by Rogers (1997) in a discussion where he rejects the f.B.m. (with a constant Hurst index) as an admissible model for the stock prices, and by Benassi & Deguy (1999) for image analysis and image synthesis (they have proposed a model with two different Hurst indexes at low and high frequencies and one frequency of change ω_c. The same kind of model is also considered in Perrin *et al.* (2001). In Bardet (2001), a statistical study of stock prices reveals different Hurst indexes at the different time scales and a theoretical study of the model proposed by Rogers (1997) is announced in Cheridito (2000).

This paper was motivated by the problem of modelization of biomecanic data, more precisely, by the regulation of the upright position of human being. Using a force platform, the position of the center of pression (C.O.P) during postural stance is determined . This position is usually measured at 100 Hz during 12 or 20 seconds or one minute, (following the experimenters) leading to set of 1500, 2000 or 6000 data[1].

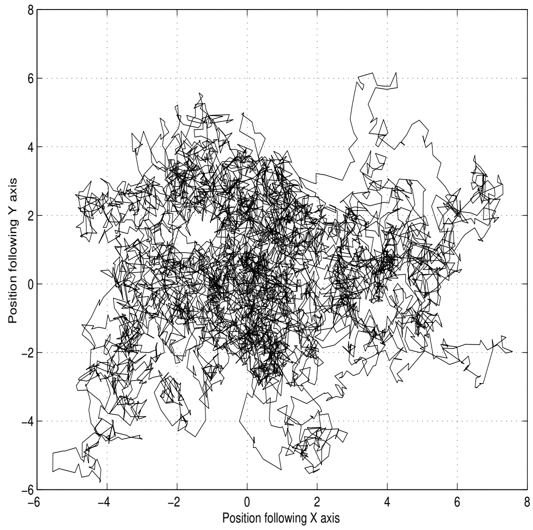


Figure 1 : An example of the trajectory of the C.O.P. during 60s at 100Hz (in mm)

During the 1970's, these data were analyzed as a set of point i.e. without taking into account their order. During the following decade some studies consider them as a process, and Collins & de Luca (1993) introduced the use of f.B.m. to modelize these data. Actually, they use a generalization of f.B.m. The position X_i of the C.O.P. is observed at times $t_i = i\Delta$ for $i = 1, \ldots, N$ ($\Delta = 0.01\ s$). The study of Collins & de Luca [13] is based on the empirical variogram

$$V(\delta) = \frac{1}{(N-\delta)} \sum_{k=1}^{N-\delta} (X_{i+\delta} - X_i)^2 \tag{1}$$

where $\delta \in \mathbb{N}^*$. For a f.B.m., we have $\mathbb{E}V(\delta) = \sigma^2\, \Delta^{2H} \times \delta^{2H}$ and after plotting the log/log graph of the variogram as a function of the lag time, i.e. $(\log \delta, \log V(\delta))$, a linear

[1]these experimental data were realized by A. Mouzat and are used in [9].

regression provides the slope $2H$. Typically, one get the following kind of figure

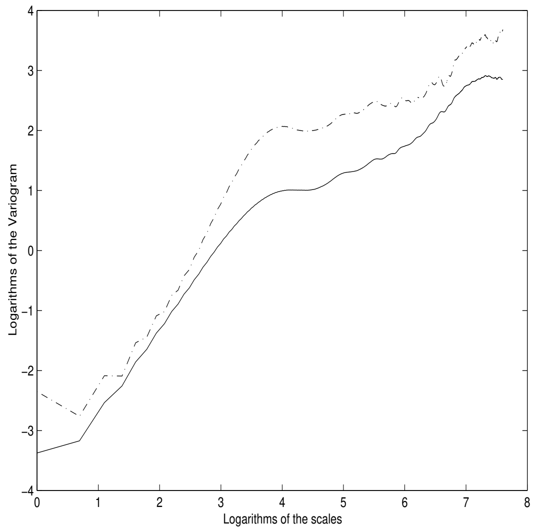


Figure 2 : An example of the log/log graph of the variogram for the previous trajectories X (−) and Y (-.)

which is considered by Collins & de Luca as a f.B.m. with two regimes *a short term* with slope $2H_0$ and *a long term* with slope $2H_1$ separated by a critical time lag δ_c and these parameters are estimated graphically. For $N = 2000$ (i.e. an observation of the COP during 20 seconds), they found $H_0 > 0.5$, $H_1 < 0.5$ and a critical time lag $\delta_c \simeq 1\ s$. These results are interpreted as corresponding to two different kinds of control of human stance : at *long term* $H_1 < 0.5$ the process is anti-persistent, at *short term* $H_0 > 0.5$ and the process is persistent or erratic : this would not lead to equilibrium.

This method has been used many times in biomecanic with different experimental conditions (opened eyes versus closed eyes, different feet angles,...), but the automatic determination of the critical time lag δ_c and of the two slopes $2H_0$, $2H_1$ is problematic for two reasons. First,the model with one change is too simple and could not correspond to the real data. In fact, the C.O.P. remains in the quadrilatere defined by the two feet, therefore the trajectories of C.O.P. are bounded, says by M. Thus we have $V(\delta) \leq 4\,M^2$ which implies $\log V(\delta) \leq 2\log(2\,M)$, for every $\delta > 0$. This induces that the slope of the variogram would converges to 0 as the lag $\delta \to +\infty$. Therefore, there should be at least another change of slope for long term. Secondly, the statistic based on variogram cannot provide convergent estimators of the model parameters. This explains the difficulties met in applications to determine the critical time lag δ_c with reliability. For these reasons, we propose a model of generalized f.B.m. with a finite number of abrupt changes on the Hurst index (following the frequencies) and develop another statistical study based on wavelet coefficients.

The following section introduces the definition and main properties of the multiscale fractional Brownian motion. A statistical study of these processes based on wavelets coefficients appears in Section 3 and leads to a test for the detection of abrupt change on the Hurst parameter. Statements of main results are displayed in Section 4. Finally, the biomecanic data are studied in Section 5. The proofs are given in appendices.

2 Description of the model

The fractional Brownian motion (f.B.m.) has been discovered by Kolmogorov (1940) and briefly studied in [19, 26]. However Mandelbrot (1965) first noticed its relevance for the modelization of natural phenomenon (Nil river). Since the seminal paper Mandelbrot et Van Ness (1968) [20], which investigates its basic properties, many other applications has been proposed. Formally, a fractional Brownian motion $B_H = \{B_H(t),\, t \in I\!\!R_+\}$ of parameters $(H,\, \sigma)$ is a real centered Gaussian process with stationary increments with $B_H(0) = 0$ and $I\!\!E\,|B_H(s) - B_H(t)|^2 = \sigma^2\,|t-s|^{2H}, \quad \forall (s,t) \in I\!\!R_+ \times I\!\!R_+$. The parameters verify $H \in]0,1[$ and $\sigma > 0$. The Hurst index H measures different properties of the f.B.m. :

1. the rate of decorrelation of the increments of B_H (short or long memory property),
2. the self-similarity of the process, i.e. for all $\lambda \in I\!\!R^+$,
$$\left(B_H(\lambda t)\right)_{t\in I\!\!R_+} \stackrel{(d)}{=} \left(\lambda^H\, B_H(t)\right)_{t\in I\!\!R_+},$$
3. the regularity of the paths, i.e. B_H is Hölder continuous of order α, $\forall\, 0 \leq \alpha < H$.

We propose Samorodnitsky & Taqqu (1994), [25, chap. 7] as a reference book on the matter. In order to define a generalization of f.B.m. and derive its main mathematical properties, we use the harmonizable representation of the f.B.m. ([25, (7.2.12), p.328], used by Kolmogorov
$$B_H(t) = \int_{I\!\!R} \frac{\left(e^{it\xi} - 1\right)}{|\xi|^{H+1/2}}\,\widehat{W}(d\xi) \tag{2}$$
where $W(dx)$ is a Brownian measure and $\widehat{W}(d\xi)$ its Fourier transform. Both the different representations of the f.B.m., i.e. the moving average [25, (7.2.4),p.321)] and the wavelet [7] representation, are less interesting for our purpose. From the harmonizable representation, a natural generalization is the multiscale fractional Brownian motion, first introduced in

Benassi & Deguy (1999), with an Hurst index depending on the frequency. More precisely we define :

Definition 2.1 *For $K \in I\!\!N$, a (M_K)-multiscale fractional Brownian motion $X_\rho = \{X_\rho(t), t \in I\!\!R_+\}$ (simplify by (M_K)-f.B.m.) is a process such that*

$$X_\rho(t) = \int_{I\!\!R} \frac{(e^{it\xi} - 1\,)}{\rho(\xi)} \overline{\widehat{W}}(d\xi) \quad \text{for } t \in I\!\!R_+ \tag{3}$$

with

- *$W(d\xi)$ is a Brownian measure and $\widehat{W}(d\xi)$ its Fourier transform in the distribution meaning.*
- *for $i = 0, \ldots, K$, there exist $(\omega_i, \sigma_i, H_i) \in (I\!\!R_+, I\!\!R_+,]0,1[)$ such that $\rho(\xi) = \sigma_i^{-1}|\xi|^{H_i+1/2}$ for $|\xi| \in [\omega_i, \omega_{i+1}[$ with $\omega_0 = 0 < \omega_1 < \ldots < \omega_K < \omega_{K+1} = +\infty$ by convention.*

The (M_K)-f.B.m. was notably introduced in order to relax the self-similarity property of f.B.m. The self-similarity is a form of invariance with respect to changes of time scale [20] and it links the behavior at high frequencies to the behavior at low frequencies. By its definition, the (M_K)-f.B.m. only verifies a self-similarity property for scales change in certain bands (in band of frequency).

Until the end of this article, we only consider a unidimensional process depending on a parameter $t \in I\!\!R_+$, but a generalization to random fields could also be done. The proof of the existence of of a continuous version of a (M_K)-f.B.m. could be found in Cramér & Leadbetter (1967). A (M_K)-f.B.m. presents K spectral changes. A (M_0)-f.B.m. is a fractional Brownian motion with a Hurst parameter H_0, and a (M_K)-f.B.m. has the same behavior that a fractional Brownian motion with a Hurst parameter H_i for frequencies $|\xi| \in [\omega_i, \omega_{i+1}[$. Therefore we could expect that a (M_K)-f.B.m. path would have the regularity H_K (at high frequencies) and a long term dependence driven by H_0 (at low frequencies). The first point is verified, but Property 2 shows that the second one is a little bit more complicated.

Property 2.1 *Let X_ρ be a (M_K)-f.B.m.*

1. *X_ρ is a Gaussian centered process with stationary increments.*
2. *For a time step $\delta > 0$, let the variogram of X_ρ be $\mathcal{V}(\delta) := I\!\!E\left(X_\rho(t+\delta) - X_\rho(t)\right)^2$. Then*

$$\mathcal{V}(\delta) \;=\; 4\sum_{i=0}^{K} \delta^{2H_i}\, \sigma_i^2 \int_{\delta\omega_i}^{\delta\omega_{i+1}} \frac{(1-\cos v)}{v^{2H_i+1}} dv. \tag{4}$$

3. *The trajectories of X_ρ are a.s. of Hölder regularity α, for every $0 \leq \alpha < H_K$.*

Now we consider the increments $Y_\rho = \{Y_\rho(t), t \in \mathbb{R}_+\}$ of X_ρ, with

$$Y_\rho(t) = X_\rho(t+1) - X_\rho(t), \quad \text{for } t \in \mathbb{R}_+.$$

Y_ρ is a centered stationary Gaussian process. Moreover, we have the following property :

Property 2.2 *Let X_ρ be a (M_K)-f.B.m. with $K \geq 1$ and Y_ρ its increments defined as previously. Define $\gamma(n) = cov\,(Y_\rho(n), Y_\rho(0))$, the correlogram of the increments of X_ρ. Then*

$$\text{if } H_0 > \frac{1}{2}, \quad \gamma(n) = \left(\frac{\pi\sigma_0^2(2H_0-1)}{\Gamma(2H_0)\sin(\pi H_0)}\right)\frac{1}{n^{2-2H_0}} + \mathcal{O}(\frac{1}{n}), \tag{5}$$

$$\text{if } H_0 \leq \frac{1}{2}, \quad \gamma(n) = \frac{P(n)}{n} + \mathcal{O}(\frac{1}{n^{2-2H_0}}), \tag{6}$$

with $P(n)$ a trigonometric polynomial depending only on $(H_i, \omega_i, \sigma_i)_{0\leq i\leq K}$ such that

$$P(n) \;=\; 8\sum_{i=1}^{K}\sin^2\left(\frac{\omega_i}{2}\right)\times\left(\frac{\sigma_{i-1}^2}{\omega_{i-1}^{2H_i-1}} - \frac{\sigma_i^2}{\omega_i^{2H_i}}\right)\times\sin(n\omega_i). \tag{7}$$

This property is interesting to study the asymptotic behavior of the dependence of the increments of X_ρ. We recall that a stationary $\mathbb{L}^2$-process $Y = \{Y(t), t \in \mathbb{N}\}$ is said to be a long-range dependent process (or long-memory process) when

$$\sum_{k=-\infty}^{\infty} |r(k)| = \infty,$$

where $r(.)$ is the covariogram of Y. Then, the long-range property of the increments of the process X_ρ depends on H_0, but not only on this value. More precisely, if $H_0 > 1/2$, the long-range (or low frequencies) behavior of Y_ρ is the same that the behavior of a fractional Brownian motion with parameter H_0. Thus, Y_ρ is a long-range dependent process. If $H_0 \leq 1/2$, the process Y_ρ is a long-range dependent process except when $(H_i, \omega_i, \sigma_i)_{0\leq i\leq K}$ verifies a relation such that the trigonometric polynomial $P(n)$ of (7) is vanished.

Remark 2.1 *If we consider the spectral density function of the stationary process Y_ρ in the meaning of Cramér & Leadbetter (1967), p. 136, i.e. the function f such that $\gamma(t) = \int_0^\infty \cos(\lambda t) f(t)dt$, we have $f(t) = \dfrac{\sin^2(t/2)}{\rho^2(t)}$. As a consequence, f is not a continuous function except if for all $i = 1, \cdots, K$, $f(\omega_i) = \lim_{t\to\omega_i^-} f(t)$, i.e. the same condition that $P(n) = 0$ for all $n \in \mathbb{N}$. Since f is not a continuous function then $\sum_{k\in\mathbb{N}} |r(k)| = \infty$ and thus Y_ρ is a long-range dependent process. For instance, if $K = 1$ and $\dfrac{\sigma_0^2}{\omega_1^{2H_0+1}} \neq \dfrac{\sigma_1^2}{\omega_1^{2H_1+1}}$, f is discontinuous in ω_1 and Y_ρ is a long-range dependent process even if $H_0 < 1/2$ and $H_1 < 1/2$.*

3 A statistical study of the multiscale fractional Brownian motion based on wavelet analysis

Let X_ρ be a (M_K)-f.B.m. defined by (3). We observed one path of the process X_ρ on the interval $[0, T_N]$ at the discrete times $t_i = i \times \Delta_N$ for $i = 1, \dots, N$ and $T_N = N \times \Delta_N$. We want to estimate the parameters of the (M_K)-f.B.m. that is K, $(H_0, H_1, \dots, H_K)$, $(\sigma_0, \sigma_1, \dots, \sigma_K)$ and $(\omega_1, \dots, \omega_K)$. We consider the asymptotic $N \to +\infty$, $\Delta_N \to 0$ and $T_N \to +\infty$.

Let's first explain the difficulties to identify the model with the variogram. Intuitively, when the time lag δ satisfies $\dfrac{1}{\omega_{i+1}} < \delta < \dfrac{1}{\omega_i}$, from (4) the main part of $\mathcal{V}(\delta)$ should be $\delta^{2H_i}\sigma_i^2 \int_{\delta\omega_i}^{\delta\omega_{i+1}} \dfrac{(1-\cos v)}{v^{1+2H_i}} dv$ which would imply

$$\mathcal{V}(\delta) \sim \delta^{2H_i}\sigma_i^2 C(H_i). \tag{8}$$

But (8) is a rough approximation which is approximately satisfied when all the quotients $\dfrac{\omega_{i+1}}{\omega_i}$ are large. For instance, when $\dfrac{\omega_{i+1}}{\omega_i} = 100$, if we chose $\delta = \dfrac{10}{\omega_i}$, formula (8) provide us a good approximation. But, when some quotients are smaller, numerical simulations show that (8) is no more verified and the log-log plot of the variogram does not exhibit any linear part. Moreover, when the model is misspecified the variogram model could lead to inadequate results, for example the following picture gives the case of a (M_2)-f.B.m. where the variogram method would detect only one frequency change.

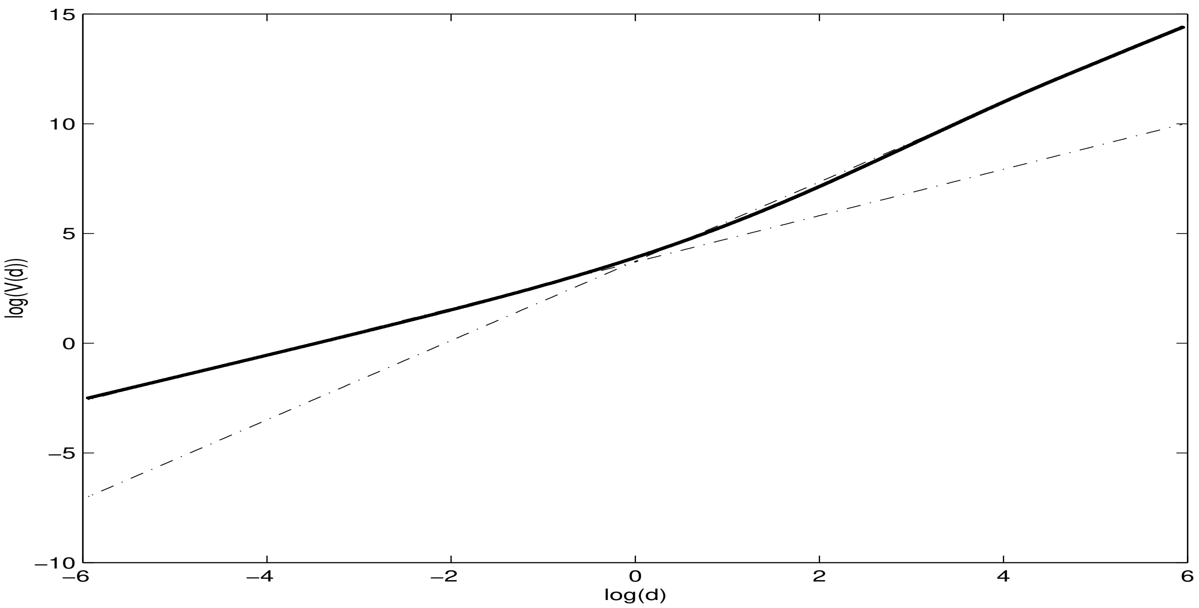


Figure 3: An example of a theoretical variogram for a (M_2)-f.B.m, with $H_0 = 0.9$, $H_1 = 0.2$, $H_2 = 0.5$, and $\sigma_0 = \sigma_1 = \sigma_2 = 5$ and $\omega_1 = 0.05$, $\omega_2 = 0.5$.

A natural idea for the identification of a (M_K)-f.B.m. is a parametric estimation. More precisely, like the increments of a (M_K)-f.B.m. form a long-memory stationary Gaussian process, the Whittle maximum likelihood estimator of the different parameters should be used. But the covariance of the increments has the asymptotic behavior described in Property 2.2 and if $H_0 < 1/2$, this behavior cannot be written as $\gamma(n) = L(n)n^{-d}$ with L a slow varying function. As a consequence we cannot apply results of Fox and Taqqu (1986), Dahlhaus (1989) or Giraïtis and Surgaïlis (1990) that are the main papers on the subject. Moreover, if $H_0 < 1/2$, the spectral density of the increments is not as regular as it is required to obtain convergence theorem for MLE.

We prefer here to use a semi-parametric statistics. We deduce from previous conclusions that an interesting method to estimate the parameters of a (M_K)-f.B.m. must be a spectral domain method. We chose a method based on a wavelet analysis (another possibility could be a log-periodogram method, but the problem of the localization of frequencies could be insuperable). This method has been introduced by Flandrin (1992) and was developed by Abry *et al.* (2001) and Bardet *et al.* (2000). Let ψ be a "mother" wavelet such that:

Assumption (A1): $\psi : \mathbb{R} \mapsto \mathbb{R}$ *is a* $\mathcal{C}^\infty$ *function verifying:*
- *for all* $m \in \mathbb{R}$, $\int_{\mathbb{R}} |t^m \psi(t)|\, dt < \infty$,
- *its Fourier transform* $\widehat{\psi}(\xi)$ *is an even function compactly supported on* $[-\beta, -\alpha] \cup [\alpha, \beta]$ *with* $0 < \alpha < \beta$.

These conditions are sufficiently mild and are satisfied by famous wavelets (in particular, Lemarié-Meyer wavelet). The admissibility property, i.e. $\int_{\mathbb{R}} \psi(t)dt = 0$ is a consequence of the second one and more generally, for all $m \in \mathbb{N}$,

$$\int_{\mathbb{R}} t^m \psi(t)dt = 0. \tag{9}$$

Note that it is not mandatory to choose ψ to be a "mother" wavelet associated to a multiresolution analysis of $\mathbb{L}^2(\mathbb{R})$ and the whole theory can be developed without resorting to this assumption : the choice of ψ is then very large.

Let $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$, define $\lambda = (a, b)$ and the family of functions ψ_λ defined by $\psi_\lambda(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t}{a} - b\right)$. Parameters a and b are so-called the scale and the shift of the wavelet transform (here we consider a continue wavelet transform). Let $d_X(a,b)$ be the wavelet coefficient of the process X for the scale a and the shift b, with

$$d_X(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t}{a} - b) X(t) dt = < \psi_\lambda, X >_{L^2(\mathbb{R})} .$$

This family of wavelet coefficients verifies the following properties :

Property 3.1 *Let ψ verify Assumption (A1) and X_ρ be a (M_K)-f.B.m. Then:*

1. *for $a > 0$, $(d_{X_\rho}(a,b))_{b\in\mathbb{R}}$ is a stationary centered Gaussian process such that :*

$$d_{X_\rho}(a,b) = \int_{\mathbb{R}} \frac{\overline{\widehat{\psi_\lambda}}(\xi)}{\rho(\xi)} \overline{\widehat{W}}(d\xi) \quad and \quad \mathbb{E}\left(d^2_{X_\rho}(a,.)\right) = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(au)\right|^2}{\rho^2(u)} du. \tag{10}$$

2. *for all $i = 0, 1, \cdots, K$, if the scale a is such that $[\frac{\alpha}{a}, \frac{\beta}{a}] \subset [\omega_i, \omega_{i+1}]$, then*

$$\mathbb{E}\left(d^2_{X_\rho}(a,.)\right) = a^{2H_i+1} \sigma_i^2 K_{H_i}(\psi), \quad with\ K_H(\psi) = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(u)\right|^2}{|u|^{2H+1}} du. \tag{11}$$

The property (11) is very interesting for the estimation of the parameters of X_ρ. In fact, we deduce that if the scale a verifies $[\frac{\alpha}{a}, \frac{\beta}{a}] \subset [\omega_i, \omega_{i+1}]$, then

$$\log\left(\mathbb{E}\left(d^2_{X_\rho}(a,.)\right)\right) = (2H_i + 1)\log a + \log \sigma_i^2 + \log K_{H_i}(\psi). \tag{12}$$

Thus, if we consider a convergent estimator of $\log\left(\mathbb{E}\left(d^2_{X_\rho}(a,.)\right)\right)$, it provides a linear model in $\log a$ and $\log \sigma_i^2$. This natural estimator is $\log I_N(a)$ with

$$I_N(a) = \frac{1}{[N/a] - 1} \sum_{k=1}^{[N/a]-1} d^2_{X_\rho}(a, k\Delta_N). \tag{13}$$

This choice is driven by the fact that only $(X_0, X_{\Delta_N}, \cdots, X_{N\Delta_N})$ is known and we suppose that ψ is essentially supported around 0. The stationarity of $d^2_{X_\rho}(a,.)$ implies that for all $a > 0$,

$$\mathbb{E}\left(I_N(a)\right) = \mathbb{E}\left(d^2_{X_\rho}(a,.)\right).$$

It is also possible to provide a multidimensional central limit theorem for $(\log I_N(a_j))_{1\le j\le m}$ and for a family of scales $(a_j)_{1\le j\le m}$.

Proposition 3.1 *Let X_ρ be a (M_K)-f.B.m., $(a_j)_{1\le j\le m} \in \mathbb{R}_+^*$ be a finite family of scales and ψ verify Assumption (A1). Denote*

$$\mathcal{I}_1(a) = \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 \rho^{-2}(a^{-1}u)\, du. \tag{14}$$

Then

$$\sqrt{N\Delta_N}\,\big(\log I_N(a_j)-\log \mathcal{I}_1(a_j)\big)_{1\leq j\leq m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m(0,\Sigma), \tag{15}$$

with $\Sigma=(s_{k,l})_{1\leq k,l\leq m}$ *a* m*-matrix such that*

$$s_{k,l}=\frac{2a_k\,a_l}{\mathcal{I}_1(a_k)\,\mathcal{I}_1(a_l)}\int_{I\!R}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a_k\xi)\widehat{\psi}(a_l\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2du. \tag{16}$$

Then, if we precise the localization of scales, *i.e.* frequencies, we obtain the interesting following consequence :

Corollary 3.1 *Let* $i\in\{0,1,\cdots,K\}$ *and assume that* $\dfrac{\beta}{\alpha}\leq\dfrac{\omega_{i+1}}{\omega_i}$*. Let* $(f_j)_{1\leq j\leq m}$ *be positive distinct real numbers (so-called frequencies) such that* $f_j\in[\dfrac{\omega_i}{\alpha},\dfrac{\omega_{i+1}}{\beta}]$ *for all* $j=1,\cdots,m$*. Under assumptions of Proposition 3.1, then*

$$\sqrt{N\Delta_N}\,\Big(\log I_N(1/f_j)+(2H_i+1)\log f_j-\log\sigma_i^2-\log K_{H_i}(\psi)\Big)_{1\leq j\leq m}\xrightarrow[N\to\infty]{\mathcal{D}}\mathcal{N}_m(0,\Sigma(i)), \tag{17}$$

with $\Sigma(i)=(s_{k,l})_{1\leq k,l\leq m}$ *a* m*-matrix such that*

$$s_{k,l}=\frac{2\,(f_k\,f_l)^{2H_i}}{K^2_{H_i}(\psi)}\int_{I\!R}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(\xi/f_k)\widehat{\psi}(\xi/f_l)}{|\xi|^{2H_i+1}}e^{-iu\xi}d\xi\right)^2du. \tag{18}$$

For N large enough, this result shows that every parameters H_i and σ_i^2 could be estimate by a linear regression of $\log I_N(1/f_j)$ by $\log f_j$, when the frequencies ω_i are known. Moreover, this central limit theorem shows that a graph of $(\log f,\,\log I_N(1/f))$ for $f>0$ exhibits different areas of asymptotic linearity : it suggests the procedure of the following section to estimate and test the frequency changes.

But before, we have to solve the discretization problem. In fact, the definition of wavelet coefficients of X_ρ needs the knowledge of a continuous path of X_ρ. But we suppose here that only a time series $(X_0,X_{\Delta_N},\cdots,X_{N\Delta_N})$ from X_ρ is provided. Thus, there is a difference between previous theoretical wavelet coefficients and empirical wavelet coefficients, the same problem that exhibits Bardet (2001). We follow the procedure and the results of this paper. So, the following empirical wavelet coefficients $e_{X_\rho}(a,b)$ have to be considered, with

$$e_{X_\rho}(a,b)=\frac{\Delta_N}{\sqrt{a}}\sum_{p=0}^{N}\psi(\frac{p\Delta_N}{a}-b)X_\rho(p\Delta_N).$$

As a consequence, we have to consider $J_N(a)$ instead of $I_N(a)$ with

$$J_N(a) = \frac{1}{[N/a]-1} \sum_{k=1}^{[N/a]-1} e^2_{X_\rho}(a, k\Delta_N). \tag{19}$$

The empirical wavelet coefficients $e_{X_\rho}(a, k\Delta_N)$ converge to theoretical one $d_{X_\rho}(a, k\Delta_N)$ when $\Delta_N \to 0$ (they are defined as a Riemann sum) and for k such that $k\Delta_N \to \infty$ and $N - k\Delta_N \to \infty$. More precisely, we have the following property:

Property 3.2 *Let ψ verify Assumption (A1) and X_ρ be a (M_K)-f.B.m. Then, for $a > 0$*

1. *for all $m \in \mathbb{N}$,*

$$\mathbb{E}\left(e^2_{X_\rho}(a, k\Delta_N)\right) = \mathbb{E}\left(d^2_{X_\rho}(a, k)\right) + \mathcal{O}\left(\Delta_N + \frac{1}{(k\Delta_N)^m} + \frac{1}{(N - k\Delta_N)^m}\right). \tag{20}$$

2. $\mathbb{E}I_N(a) = \mathbb{E}J_N(a) + \mathcal{O}(\Delta_N)$

Remark 3.1 *When $H > 1/2$, we could approximate $d(a, b)$ by the trapezoid method which would lead in Property 3.2.2 to an error in $\mathcal{O}(\Delta_N^2)$.*

Now, it is possible to provide the multidimensional central limit theorem for $(\log I_N(a_j))_{1\le j\le m}$ computed from $(X_0, X_{\Delta_N}, \cdots, X_{N\Delta_N})$:

Proposition 3.2 *Under assumptions of Proposition 3.1 and Δ_N be such that $N\Delta_N \to \infty$ when $N \to \infty$ and $N(\Delta_N)^3 \to 0$ when $N \to \infty$. Then, with the same matrix Σ than in (15),*

$$\sqrt{N\Delta_N}\left(\log J_N(a_j) - \log \mathcal{I}_1(a_j)\right)_{1\le j\le m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m(0, \Sigma). \tag{21}$$

Let $i \in \{0, 1, \cdots, K\}$ and assume that $\frac{\beta}{\alpha} \le \frac{\omega_{i+1}}{\omega_i}$. If the positive distinct real numbers, i.e. the frequencies, $(f_j)_{1\le j\le m}$ are such that $f_j \in [\frac{\omega_i}{\alpha}, \frac{\omega_{i+1}}{\beta}]$ for all $j = 1, \cdots, m$, then, with the same matrix $\Sigma(i)$ than in (18),

$$\sqrt{N\Delta_N}\left(\log J_N(1/f_j) + (2H_i+1)\log f_j - \log \sigma_i^2 - \log K_{H_i}(\psi)\right)_{1\le j\le m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m(0, \Sigma(i)). \tag{22}$$

A consequence of the discretization problem is that the convergence rate of the central limit theorem (21) that is $\sqrt{N\Delta_N}$ and thus the maximum convergence rate is $o(N^{1/3})$ from the previous conditions on Δ_N.

4 Detection of frequency changes

To detect frequency changes on the parameters we adapt the method proposed in Bertrand (2000). This article is devoted to the detection of change on the mean of independent Gaussian random variables. This method is based on the behavior of the difference of the estimator of the parameter onto the sliding right and left box. Here we should use a slight modification due to the existence of a transition zone at the vicinity of each frequency change, i.e. between $\log(\omega_i) - \log(\beta)$ and $\log(\omega_i) - \log(\alpha)$.

First, let us describe the method at an heuristic level. From Proposition 3.2, Formula (22), we have

$$\log J_N(1/f) = -(2H_i + 1) \times \log(f) + \log\left(\sigma_i^2\right) + \log\left(K_{H_i}(\psi)\right) + \varepsilon_f^{(N)}, \tag{23}$$

for the frequencies f which satisfy the condition

$$\log(\omega_i) - \log(\alpha) \leq \log(f) \leq \log(\omega_{i+1}) - \log(\beta). \tag{24}$$

Moreover we have $(N\Delta_N)^{1/2}\left(\varepsilon_{f_j}^{(N)}\right)_{1\leq j\leq m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m(0,\Sigma)$.

Formula (23) and condition (24) mean that for $\log(f) \in [\log(\omega_i) - \log(\alpha), \log(\omega_{i+1}) - \log(\beta)]$, we have a linear regression of $\log J_N(1/f)$ onto $\log(f)$ with slope $-(2H_i+1)$ and intercept $\log\sigma_i^2 + \log K_{H_i}(\psi)$ and for $\log(f) \in [\log(\omega_{i+1}) - \log(\alpha), \log(\omega_{i+2}) - \log(\beta)]$ a linear regression with slope $-(2H_{i+1}+1)$ and intercept $\log\sigma_{i+1}^2 + \log K_{H_{i+1}}(\psi)$. We have a problem of detection of abrupt change on the parameter of a linear regression (as in [10]), but with a transition zone for $\log(f) \in]\log(\omega_{i+1}) - \log(\beta),\ \log(\omega_{i+1}) - \log(\alpha)[$. As a consequence, we take the difference between the estimators of the regression parameters onto a right and a left boxes of the same size, but with a hole between the two boxes to avoid the transition zone (at the change point).

Remark 4.1 *Condition (24) implies that* $\omega_{i+1} > \dfrac{\beta}{\alpha} \times \omega_i$*. Therefore we could only detect the frequency changes sufficiently spaced. For instance, if we choose the Lemarié-Meyer wavelet, we get* $\beta/\alpha = 4$ *which leads to the condition* $\omega_{i+1} > 4 \times \omega_i$*.*

We now give the precise definitions and assumptions. For simplification, we suppose that the logarithms of the chosen frequencies are constantly spaced and we only study the detection of changes on the Hurst parameter[2]. More precisely, we assume the following notations and conditions :

[2] Detection and identification of the Hurst parameter and the variance will be study in forthcoming work.

1. Let $[\omega_{min}, \omega_{max}]$ the frequency band ($\omega_{min} > 0$) where we would detect frequency changes.
2. Let $q > 1$ be the frequency ratio between two successive frequencies, defined by
$$q = \left(\frac{\beta}{\alpha}\right)^{1/\kappa} \quad i.e. \quad \kappa \log q = \log(\beta/\alpha) \quad \text{with } \kappa \in \mathbb{N}^*. \tag{25}$$
with $\kappa \in \mathbb{N}^*$ such that $\kappa - 1$ is the number of chosen frequencies between the left and the right boxes.
3. Let $\Lambda + 1$ be the number of chosen frequencies in a box applied for the regression. The integer Λ satisfies $\Lambda \geq 1$ and the following condition which insures that two different frequency changes could be detected separately,
$$(2\Lambda + \kappa) \times \log q \quad \leq \quad \min_{\omega_i, \omega_{i+1} \in [f_0, f_M]} \left[\log(\omega_{i+1}) - \log(\omega_i)\right] \tag{26}$$
with f_0, f_M defined by (28) below. This condition is equivalent to
$$2\Lambda \leq \kappa \times \left\{ \min_{\omega_i, \omega_{i+1} \in [f_0, f_M]} \left[\frac{\log(\omega_{i+1}) - \log(\omega_i)}{\log(\beta/\alpha)}\right] - 1 \right\}. \tag{27}$$
4. The chosen frequencies are $(f_\nu)_{0 \leq \nu \leq M}$ with :
$$\begin{aligned} f_\nu = q^\nu \times f_0 \qquad & \text{for } \nu = 0, 1, \cdots, M \text{ where } \; f_0 = \frac{\omega_{min}}{\beta} \times q^{-\Lambda} \\ & \text{and } M = (2\,\Lambda + \kappa) + \kappa \left[\frac{\log(\omega_{max}/\omega_{min})}{\log(\beta/\alpha)}\right], \end{aligned} \tag{28}$$
because $f_M \geq \dfrac{\omega_{max}}{\alpha} q^\Lambda$.

We define now the test statistic. For two integers k, l such that $0 \leq k < l \leq M$, we define the following matrices :
$$\mathbf{A}_{[k,l]} = \begin{pmatrix} \log f_k & 1 \\ \log f_{k+1} & 1 \\ \vdots & \vdots \\ \log f_l & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}^{(N)}_{[k,l]} = \begin{pmatrix} \log(J_N(1/f_k)) \\ \log(J_N(1/f_{k+1})) \\ \vdots \\ \log(J_N(1/f_l)) \end{pmatrix}.$$
Then, the test statistic is $D^{(N)}_1(\Lambda, \mu)$ with
$$\begin{aligned} D^{(N)}(\Lambda, \mu) = \begin{pmatrix} D^{(N)}_1(\Lambda, \mu) \\ D^{(N)}_2(\Lambda, \mu) \end{pmatrix} \quad = \quad & \frac{1}{2}\Big[\left(\mathbf{A}'_{[\mu,\mu+\Lambda]}\mathbf{A}_{[\mu,\mu+\Lambda]}\right)^{-1}\mathbf{A}'_{[\mu,\mu+\Lambda]}\mathbf{Y}^{(N)}_{[\mu,\mu+\Lambda]} \\ & - \left(\mathbf{A}'_{[\mu-\kappa-\Lambda,\mu-\kappa]}\mathbf{A}_{[\mu-\kappa-\Lambda,\mu-\kappa]}\right)^{-1}\mathbf{A}'_{[\mu-\kappa-\Lambda,\mu-\kappa]}\mathbf{Y}^{(N)}_{[\mu-\kappa-\Lambda,\mu-\kappa]}\Big] \end{aligned} \tag{29}$$

for $\Lambda+\kappa \leq \mu \leq M-\Lambda$. The test statistic $D_1^{(N)}(\Lambda,\mu)$ corresponds to the difference of the estimation of H between the right and the left boxes with a hole of $\kappa-1$ frequencies.

Then we provide bounds of the error probabilities of the first and second type - "false alarm probability" and "non detection probability". When two frequency changes are sufficiently spaced (i.e. satisfy the condition (26)), there appear separately on $\mu \mapsto D_1^{(N)}(\Lambda,\mu)$. Therefore, for notational convenience, we restrict us to the case of one frequency change in the frequency band $[\omega_{min},\omega_{max}]$. Let us precise the assumptions. We want to test the null hypothesis

(H0) $\quad K=0$, *i.e.* $\varrho(\xi)=\sigma_0^{-1}|\xi|^{H_0+1/2}\quad$ for $\xi \in [\omega_{min},\omega_{max}]$,

against the alternative hypothesis

(H1) $\quad K=1$, *i.e.* $\begin{array}{ll} \varrho(\xi)=\sigma_0^{-1}|\xi|^{H_0+1/2} & \text{for } \omega_{min} \leq \xi < \omega_1, \\ \varrho(\xi)=\sigma_1^{-1}|\xi|^{H_1+1/2} & \text{for } \omega_1 \leq \xi \leq \omega_{max}. \end{array}$

We fix a threshold $\eta>0$ and detect a frequency change, if

$$\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda} |D_1^{(N)}(\Lambda,\mu)| > \eta. \tag{30}$$

For a bilateral test, we have

$$\begin{array}{ll} \mathbb{P}_{(H0)}\Big(\max\limits_{\Lambda+\kappa\leq\mu\leq M-\Lambda}\Big|D_1^{(N)}(\Lambda,\mu)\Big|>\eta\Big) & \text{is the error probability of type 1} \\ \mathbb{P}_{(H1)}\Big(\max\limits_{\Lambda+\kappa\leq\mu\leq M-\Lambda}\Big|D_1^{(N)}(\Lambda,\mu)\Big|<\eta\Big) & \text{is the error probability of type 2.} \end{array}$$

We get the following bounds :

Proposition 4.1 (Probability of false alarm) *Let ψ verify Assumption (A1) with $\alpha>1$, X_ρ be a (M_K)-f.B.m. and Δ_N be such that $N\Delta_N \to \infty$ and $N(\Delta_N)^3 \to 0$ when $N\to\infty$. Then for every frequency band $[\omega_{min},\,\omega_{max}]$, every positive integers κ, Λ satisfying (27) and every $\eta>0$, we have for N large enough*

$$\mathbb{P}_{(H0)}\Big(\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda}\Big|D_1^{(N)}(\Lambda,\mu)\Big|>\eta\Big) \leq \frac{1}{N\Delta_N}\times\frac{C_1}{\eta^2}\times\left(1+\kappa\left[\frac{\log\left(\omega_{max}/\omega_{min}\right)}{\log(\beta/\alpha)}\right]\right), \tag{31}$$

with the constant C_1 such that

$$C_1 = 72\left(\frac{\kappa}{\Lambda}\right)^3\left(\frac{\beta}{\alpha}\right)^{2\Lambda/\kappa}\left(\frac{\beta}{\omega_{min}}\right)^2\frac{L(\psi)}{\log^2(\beta/\alpha)} \quad \textit{and} \quad L(\psi)=\frac{\int_\alpha^\beta\left|\widehat{\psi}(v)\right|^4 v^{-2}\,dv}{\left(\int_\alpha^\beta\left|\widehat{\psi}(v)\right|^2 v^{-3}\,dv\right)^2}. \tag{32}$$

Proposition 4.2 (Probability of non detection) *Let ψ verify Assumption (A1) with $\alpha > 1$, X_ρ be a (M_K)-f.B.m. and Δ_N be such that $N\Delta_N \to \infty$ and $N(\Delta_N)^3 \to 0$ when $N \to \infty$. Assume that the hypothesis (H1) is satisfied with a frequency change $\omega_1 \in [\omega_{min}, \omega_{max}]$, and*

$$|H_1 - H_0| \geq d_0 \quad \textit{with} \quad d_0 > 0. \tag{33}$$

Then for every positive integers κ, Λ satisfying (27), every $\eta > 0$ such that $d_0 > \widetilde{\eta}$, where

$$\widetilde{\eta} = \eta + \frac{3}{(\Lambda+1)(\Lambda+2)} \times \left(3 + 2\kappa \frac{|\log(\sigma_1/\sigma_0)|}{\log(\beta/\alpha)}\right), \tag{34}$$

we have for N large enough, and with C_1 defined by (32),

$$\mathbb{P}_{(H1)}\left(\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda} |D_1^{(N)}(\Lambda,\mu)| < \eta\right) \leq \frac{1}{N\Delta_N} \times \frac{C_1}{(d_0-\widetilde{\eta})^2}. \tag{35}$$

Discussion about the choice of parameters

How to chose the different parameters κ, Λ, η and d_0 in the case of the previous test (one frequency change) ? First, α, β and $\omega_{min}, \omega_{max}$ are known; for the Lemarié-Meyer wavelet (used in the following simulations), we have $\beta/\alpha = 4$. Assume now that N and Δ_N are such that N and $N\Delta_N$ are large enough and $\Delta_N = o(N^{1/3})$.

Choice of κ and Λ : There are 2 criterions for the choice of integers κ and Λ.

1. First, when we consider N like a large but finite number, calculation considerations imply $[N f_0] \geq 2$ and $f_M \leq 1/\Delta_N$. As a consequence, we obtain a bound for the ratio $\frac{\Lambda}{\kappa}$:
$$\frac{\Lambda}{\kappa} \leq \frac{1}{\log(\beta/\alpha)} \min\left\{\log\left(\frac{N\,\omega_{min}}{2\beta}\right) \ , \ \log\left(\frac{\alpha}{\Delta_N\,\omega_{max}}\right)\right\}.$$
2. Second, κ and Λ have to minimize the bound of the error probability of type 1 and 2. As a consequence, the constant C_1 has to be close to its minimum obtained for
$$\frac{\Lambda}{\kappa} \simeq \frac{3}{2\log(\beta/\alpha)}.$$

Choice of η and d_0 : The value of η is fixed from the chosen confidence level of the error probability α_0 of type 1 (for instance $\alpha_0 = 0.05$). Then, $\widetilde{\eta}$ can be computed when we assume the knowledge of an upper bound of $|\log(\sigma_1/\sigma_0)|$. Finally, for every level β_0 of the the error probability of type 2 we obtain the minimum value d_0 of detectable frequency changes.

5 Application in Biomecanic

The force-plate was an AMTI model OR6-5. The signals were amplified through an AMTI model SGA6-4 amplifier. The numerical-analog converter card was a Data Translation model DT2801 (12 bits). Recorded data were the three forces and the three moments of the efforts of contact on the surface of the force-plate. From the forces and moments measured on the platform, we calculated the COP coordinates (X,Y) of the subject in the force-plate referential. X axis of the platform corresponds to the fore-aft direction and Y axis corresponds to the medio-lateral direction, whereas Z axis corresponds to the vertical axis (positive downwards).The rate frequency was 100 hertz for one duration of 60 seconds. The subjects were placed on the force-plate in a cabin conformed to the standards of the Association Française de Posturologie (AFP). They had to stand upright during one minute. They had to fix a plumb line placed at about 90 centimeters in front of them. The subjects' feet positions were controlled by a special frame fixed on the force-plate with a clearance of 2 cm between the feet and an angle of 15^0 between the feet.

We apply our statistics to different trajectories with the following parameters :

- $N = 6000$ and $\Delta_N = 1/20$.
- The mother wavelet is the Lemarié-Meyer wavelet, and $\alpha = 2\pi/3$, $\beta = 8\pi/3$.
- We want to detect a frequency change in the band $[\omega_{min}, \omega_{max}] = [0.5, 2]$ Hz.
- $\kappa = 200$ and $\Lambda = 300$. As a consequence, $M = 1000$, $f_0 \simeq 0.0075$Hz and $f_M \simeq$ 7.64Hz.

The following figures exhibit the differences between our statistics for the Y-trajectory and a trajectory of a f.B.m. (obtained from a simulation with the circulant matrix algorithm) with a Hurst parameter $H = 0.3$.

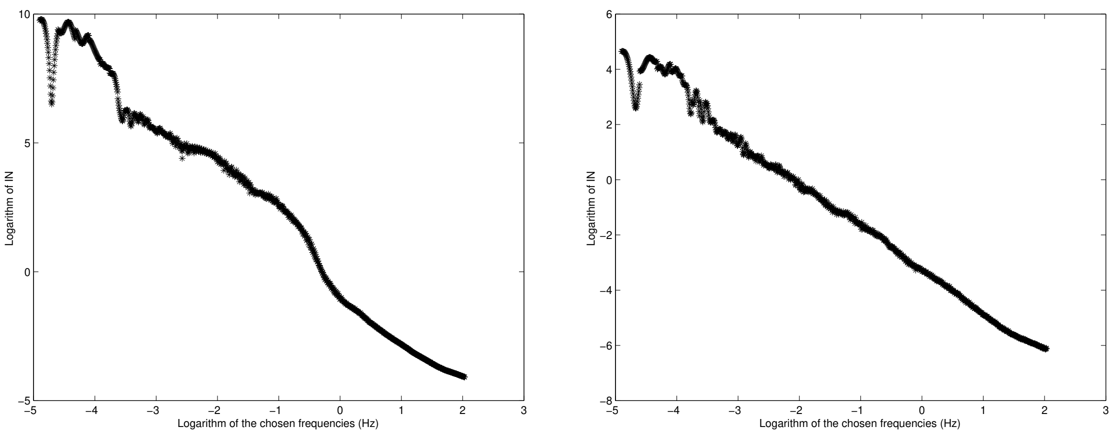
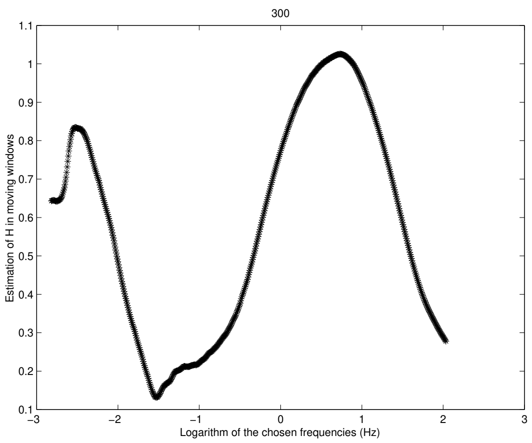


Figure 4 : The double logarithm plotting of the variance of wavelet coefficients (left,

Y-trajectory, right, f.B.m. trajectory)

In figure 4, we observe the linearity of the graph for the f.B.m trajectory for frequencies in a band $\simeq [0.03, 7.5]$ Hz, and two different bands of linearity ($\simeq [0.75, 7.5]$ Hz and $\simeq [0.03, 0.2]$ Hz) for the Y-trajectory, those two bands are spaced by a frequency band of length $\simeq 2\log 2$ Hz as expected by theoretical results. We also observe in both graphs the important irregularity for low frequencies (less than 0.03 Hz) explained by the fact that $N \times f$ is too small to compute an estimation of the variance of wavelet coefficients.
We obtain more details with the two following figures. Figure 5 exhibits the different values of the estimation of H following the logarithm of the larger frequency of the regression. As a conclusion, we obtain a mean value $\simeq 0.28$ of H for the f.B.m. trajectory in the linearity band, and 2 different values of H for the Y-trajectory, $\simeq 0.3$ for the band $[0.75, 7.5]$ Hz and $\simeq 0.2$ for the band $[0.03, 0.2]$ Hz.



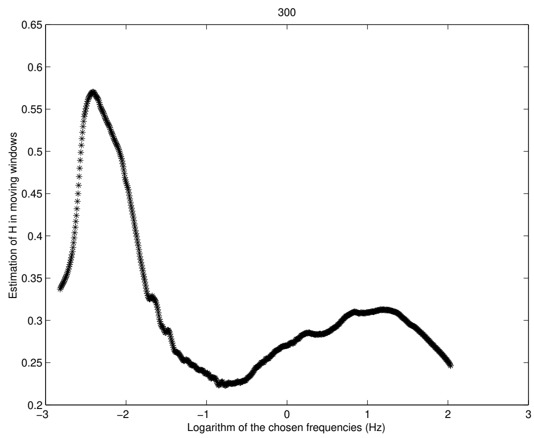


Figure 5 : Estimation of H in the moving window of length $\Lambda + 1$ frequencies in function of the logarithm of frequencies (left, Y-trajectory, right, f.B.m. trajectory)

Figure 6 provides the behavior of the test statistic D_1 following the logarithm of the larger frequency. Finally, we deduce that the Y-trajectory may be modelized by a M_1-f.B.m. with a frequency change $\omega_1 \simeq 0.5$ Hz.

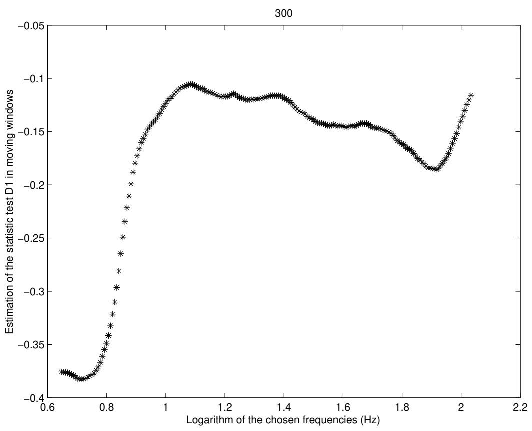


Figure 6 : The statistic test D_1 in function of the logarithm of frequencies (left, Y-trajectory, right, f.B.m. trajectory)

As a conclusion, the following figure 7 provides a comparison of the previous graphs and the graphs of the X-trajectory and Y-trajectory with a clearance of 2 cm between the feet. We find the same behavior for the X-trajectory that for the Y-trajectory (a M_1-f.B.m. with a frequency change $\omega_1 \simeq 0.5$ Hz) but with different values for H : $H \simeq 0.45$ for low frequencies and $H \simeq 0.35$ for hight frequencies. For the Y-trajectory with a clearance of 2 cm between the feet, we find the same behavior that without the clearance but the difference between both the values of H is smaller ($H \simeq 0.27$ for low frequencies and $H \simeq 0.33$ for high frequencies) : a clearance between seems to stabilize the upright position around a unique behavior.

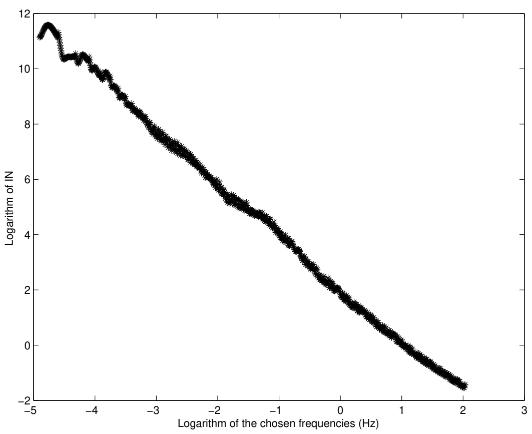


Figure 7 : The double logarithm plotting of the variance of wavelet coefficients (left, X-trajectory, right, Y-trajectory with a clearance of 2 cm between the feet)

Remark 5.1 *Unfortunately, it is not possible to apply the test of the frequency change detection to these data. In fact, the chosen parameters lead to a very large and unusable threshold* η*. There are different reasons for this difficulty. First, the concrete problem with a too small coefficient* $N\Delta_N$ *and a too large ratio* $\omega_{max}/\omega_{min}$*. Second, the utilization of the Lemarié-Meyer wavelet with a two large ratio* β/α*. This difficulty could be avoided by the construction of a mother wavelet with a smaller Fourier support. Third, the overestimations of both the test error probabilities are insufficiently sharp. For instance, with the Lemarié-Meyer wavelet,* $L(\psi) \simeq 250$ *is a too large constant imposed by its independence in term of* H*.*

A Proofs

A.1 Proofs of section 2

Proof. [**Property 2.1**]

1. See Cramér & Leadbetter (1967) for the proof of the stationarity of the increments of X_ρ.
2. We easily obtain :
$$\begin{aligned} \mathcal{V}(\delta) &= \int_{I\!R} \frac{|e^{i\delta\xi}-1|^2}{\rho^2(\xi)} d\xi \;=\; 4\int_0^{+\infty} \frac{(1-\cos(\delta\xi))}{\rho^2(\xi)} d\xi \\ &= 4\sum_{i=0}^{K} \sigma_i^2 \int_{\omega_i}^{\omega_{i+1}} \frac{(1-\cos(\delta\xi))}{\xi^{2H_i+1}} d\xi, \end{aligned}$$
that provides us (4) after the change of variable $\delta\xi = v$.
3. First, we need the asymptotic behavior of the variogram when the time lag $\delta \to 0$. We have :
$$\mathcal{V}(\delta) = \delta^{2H_K}\,\sigma_K^2\,C(H_K) + \mathcal{O}(\delta^2) \quad \text{when} \quad \delta \to 0, \tag{36}$$
where $C(H) = 4\int_0^{+\infty} [1-\cos(v)] \times v^{-(2H+1)}\,dv$. In fact, from (4) we get
$$\begin{aligned} \mathcal{V}(\delta) &= \delta^{2H_K}\,\sigma_K^2\,C(H_K) \;-\; \delta^{2H_K}\,\sigma_K^2 \int_0^{\delta\omega_K} [1-\cos(v)] \times v^{-(2H_K+1)}\,dv \\ &+ \sum_{i=0}^{K-1} \left\{ \delta^{2H_i}\,\sigma_i^2 \int_{\delta\omega_i}^{\delta\omega_{i+1}} [1-\cos(v)] \times v^{-(2H_i+1)}\,dv \right\} \\ &+ \delta^{2H_0}\,\sigma_0^2 \int_0^{\delta\omega_0} [1-\cos(v)] \times v^{-(2H_0+1)}\,dv. \end{aligned}$$

When $\delta \to 0$

$$\begin{aligned}\int_0^{\delta\omega} [1-\cos(v)] \times v^{-(2H+1)}\, dv &\simeq \frac{1}{2}\int_0^{\delta\omega} v^{2-(2H+1)}\, dv \\ &= \delta^{2(1-H)} \times \frac{\omega^{2(H-1)}}{4(H-1)}\end{aligned}$$

which implies

$$\delta^{2H}\int_0^{\delta\omega} [1-\cos(v)] \times v^{-(2H+1)}\, dv = \mathcal{O}(\delta^2)$$

and after

$$\delta^{2H}\int_{\delta\omega_i}^{\delta\omega_{i+1}} [1-\cos(v)] \times v^{-(2H+1)}\, dv = \mathcal{O}(\delta^2)$$

which induces (36). From asymptotic behavior, see (36), and with $D > 0$, we get

$$I\!\!E\left[|X_\rho(t+\delta) - X_\rho(t)|^2\right] = D \times \delta^{2H_K} + \mathcal{O}(\delta^2).$$

Therefore there exists a real $T_1 > 0$ such that for all $s, t \in [0, T_1[$

$$I\!\!E\left[|X_\rho(t) - X_\rho(s)|^2\right] \le D \times |t-s|^{2H_K}.$$

Since the increments $X_\rho(t) - X_\rho(s)$ are Gaussian r.v., for every integer $p > 0$ there exits a constant $C_p > 0$ such that for all $s, t \in [0, T_1[$

$$I\!\!E\left[|X_\rho(t) - X_\rho(s)|^{2p}\right] \le D \times |t-s|^{2p\,H_K}.$$

From the Kolmogorov Theorem (see for example [23, p.26]), the process X_ρ have a modification with continuous paths which are Hőlderian of order α, for all $\alpha < H_K - 1/(2p)$. Since this is true for every integer p, we deduce that X_ρ have a modification with continuous paths which are Hőlderian of order α, for all $\alpha < H_K$. We get the same result on each interval $[k\,T_1,\,(k+1)\,T_1[$ with $k \in \mathbf{Z}$ and we deduce the existence of a modification of X_ρ with continuous paths which are almost surely Hőlderian of order α, for all $\alpha < H_K$. ■

Proof. [**Property 2.2**] First, we have

$$\gamma(n) = 8\sum_{i=0}^{K} \sigma_i^2 \int_{\omega_i}^{\omega_{i+1}} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_i+1}}\, d\xi. \tag{37}$$

Indeed, using (3) and the definition of $\gamma(n)$, we get :

$$\begin{aligned}
\gamma(n) = \mathrm{cov}\,(Y_\rho(n), Y_\rho(0)) &= \mathrm{cov}\,[(X_\rho(n+1) - X_\rho(n)), (X_\rho(1) - X_\rho(0))] \\
&= \int_{I\!R} \frac{(e^{i\xi(n+1)} - e^{i\xi n})}{\rho(\xi)} \frac{(e^{-i\xi} - 1)}{\rho(\xi)}\, d\xi \\
&= \int_{I\!R} e^{i\,n\xi} \frac{\left|e^{-i\xi} - 1\right|^2}{\rho^2(\xi)}\, d\xi \\
&= \int_0^{+\infty} \frac{\cos(n\xi)\sin^2(\xi/2)}{\rho^2(\xi)}\, d\xi,
\end{aligned}$$

which induces (37) by definition of the (M_k)-f.B.m. Formulae (5) and (6) are asymptotic expansion of $\gamma(n)$ following the powers of $1/n$. After two integrations by parts, we have the following calculations : for every $\omega > 0$ and $H \in]0,1[$

$$\begin{aligned}
&\int_\omega^\infty \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H+1}} d\xi = \\
&= \frac{1}{n}\left[\frac{-\sin(n\omega)\sin^2(\omega/2)}{\omega^{2H+1}} - \int_\omega^\infty \sin(n\xi)\left(\frac{\sin\xi}{2\xi^{2H+1}} - \frac{(2H+1)\sin^2(\xi/2)}{\xi^{2H+2}}\right) d\xi\right] \\
&= -\frac{1}{n}\left(\frac{\sin(n\omega)\sin^2(\omega/2)}{\omega^{2H+1}}\right) - \frac{1}{n^2}\cos(n\omega)\left(\frac{\sin\omega}{\omega^{2H+1}} - (2H+1)\frac{\sin^2(\omega/2)}{\omega^{2H+2}}\right) \\
&\quad + \frac{1}{n^2}\int_\omega^\infty \cos n\xi \left(\frac{\cos\xi}{\xi^{2H+1}} - (2H+1)\frac{\sin(\xi)}{\xi^{2H+2}} + (2H+1)(2H+2)\frac{\sin^2(\xi/2)}{\xi^{2H+2}}\right) d\xi
\end{aligned}$$

For $\omega > 0$ and $H > 0$, the last integral is bounded by

$$F \times \int_\omega^\infty \left[\xi^{-(2H+1)} + \xi^{-(2H+2)}\right] \le F \times \left[\omega^{-2H} + \omega^{-(2H+1)}\right] < +\infty,$$

with $F > 0$. This induces for every $\omega > 0$ and $H \in]0,1[$

$$\int_\omega^\infty \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H+1}} d\xi = -\frac{1}{n}\left(\frac{\sin(n\omega)\sin^2(\frac{\omega}{2})}{\omega^{2H+1}}\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{38}$$

We immediately deduce from (38) that for $i = 1, 2, .., K$, with $\omega_{K+1} = \infty$, we have

$$\int_{\omega_i}^{\omega_{i+1}} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_i+1}} d\xi = \tag{39}$$

$$= \frac{1}{n}\left(\frac{\sin(n\omega_{i+1})\sin^2(\frac{\omega_{i+1}}{2})}{\omega_{i+1}^{2H_i+1}} - \frac{\sin(n\omega_i)\sin^2(\frac{\omega_i}{2})}{\omega_i^{2H_i+1}}\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{40}$$

¿From the other hand, consider the first term in the sum in the right hand side of (37). We have

$$\int_0^{\omega_1} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}} d\xi =$$

$$= \int_0^{+\infty} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}} d\xi - \int_{\omega_1}^{+\infty} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}} d\xi. \tag{41}$$

The first integral correspond to the case of a constant Hurst parameter, that is the usual f.B.m. Let X_0 be a f.B.m. with parameter H_0 and variance 1, defined by

$$X_0(t) = \frac{1}{C(H_0)} \int_{I\!R} \frac{(e^{it\xi}-1)}{|\xi|^{H_0+1/2}} \widehat{W}(d\xi) \quad \text{for } t \in I\!R_+$$

with $C(H_0) = \pi^{1/2}(H_0 \Gamma(2H_0)\sin(\pi H_0))^{-1/2}$, then we have

$$\mathrm{cov}\left((X_0(n+1)-X_0(n), X_0(1)-X_0(0)\right) = \frac{1}{C^2(H_0)} \int_0^\infty \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}} d\xi$$

and (see [25] p. 335) we have the following expansion

$$\mathrm{cov}\left((X_0(n+1)-X_0(n), X_0(1)-X_0(0)\right) = H_0(2H_0-1)\frac{1}{n^{2-2H_0}} + \mathcal{O}(\frac{1}{n^{3-2H_0}}).$$

Combined with (38) and (41), for all $H_0 \in]0,1[$, we have

$$\begin{aligned}\int_0^{\omega_1} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}} d\xi &= \\ = \left(\frac{\pi(2H_0-1)}{\Gamma(2H_0)\sin(\pi H_0)}\right)\frac{1}{n^{2-2H_0}} &+ \left(\frac{\sin(n\omega_1)\sin^2(\frac{\omega_1}{2})}{\omega_1^{2H_0+1}}\right)\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}+\frac{1}{n^{3-2H_0}}\right). \quad (42)\end{aligned}$$

As a consequence from (39) and (42), for $H_0 > 1/2$, we obtain (5) and if $0 < H_0 \leq 1/2$,

$$\begin{aligned}&\mathrm{cov}\left(Y_\rho(n)Y_\rho(0)\right) \\ &= \frac{8}{n}\left[\sum_{i=1}^{K-1}\sigma_i^2\left(\frac{\sin(n\omega_{i+1})\sin^2(\frac{\omega_{i+1}}{2})}{\omega_{i+1}^{1+2H_{i+1}}} - \frac{\sin(n\omega_i)\sin^2(\frac{\omega_i}{2})}{\omega_i^{1+2H_i}}\right)\right. \\ &\quad \left. +\sigma_0^2\left(\frac{\sin(n\omega_1)\sin^2(\frac{\omega_1}{2})}{\omega_1^{2H_0+1}}\right) - \sigma_K^2\frac{\sin(n\omega_K)\sin^2(\frac{\omega_K}{2})}{\omega_K^{2H_K+1}}\right] \\ &\quad + \frac{\pi\sigma_0^2(2H_0-1)}{\Gamma(2H_0)\sin(\pi H_0)}\frac{1}{n^{2-2H_0}} + \mathcal{O}(\frac{1}{n^2}),\end{aligned}$$

and (6) is proved. ■

A.2 Proofs of section 3

Proof. [**Property 3.1**] First we have

$$\begin{aligned} d_{X_\rho}(\lambda) &= \int_{I\!R}\psi_\lambda(t)\left[\int_{I\!R}\frac{\left(e^{it\xi}-1\right)}{\rho(\xi)}\widehat{W}(d\xi)\right]dt \\ &= \int_{I\!R}\frac{1}{\rho(\xi)}\left[\int_{I\!R}\left(e^{it\xi}-1\right)\psi_\lambda(t)dt\right]\widehat{W}(d\xi)\end{aligned}$$

But ψ verifies $\int \psi = 0$ and it implies (10). Now, we prove that $(d_{X_\rho}(a,b))_{b\in\mathbb{R}}$ is a stationary centered Gaussian process. The only problem is the stationarity. It is obvious that

$$\overline{\widehat{\psi_\lambda}}(\xi) = \sqrt{a}e^{-iab\xi}\overline{\widehat{\psi}}(a\xi), \tag{43}$$

Combined with (10), we deduce that for $a > 0$ and $(b, b') \in \mathbb{R}^2$, we have:

$$I\!E\left(d_{X_\rho}(a,b)d_{X_\rho}(a,b')\right) = \int_{\mathbb{R}} e^{-ia(b-b')\xi}\left|\frac{\widehat{\psi}(a\xi)}{\rho(\xi)}\right|^2 d\xi.$$

Thus, for a given $a > 0$, $I\!E\left(d_{X_\rho}(a,b)d_{X_\rho}(a,b')\right)$ is only depending on $(b-b')$ which induces that $(d_{X_\rho}(a,b))_{b\in\mathbb{R}}$ is a stationary process.

The second part of the property is a consequence of this first part. In fact,

$$\begin{aligned} I\!E\left(d^2_{X_\rho}(a,.)\right) &= \int_{\mathbb{R}}\left|\frac{\widehat{\psi_\lambda}(\xi)}{\rho(\xi)}\right|^2 d\xi \\ &= a\int_{\mathbb{R}}\left|\frac{\widehat{\psi}(av)}{\rho(v)}\right|^2 dv \quad \text{(from a change of variable).} \end{aligned}$$

But the scale a verifies $[\frac{\alpha}{a}, \frac{\beta}{a}] \subset]\omega_i, \omega_{i+1}[$ and ψ verifies Assumption (A1). Thus

$$\begin{aligned} I\!E\left(d^2_{X_\rho}(a,.)\right) &= 2a\int_{\alpha/a}^{\beta/a}\left|\frac{\overline{\widehat{\psi}}(av)}{\rho(v)}\right|^2 dv \\ &= 2a\int_{\alpha/a}^{\beta/a}\left|\widehat{\psi}(a\,v)\right|^2 \sigma_i^2\, v^{-(2H_i+1)}dv \\ &= a\sigma_i^2\int_{\mathbb{R}}\left|\widehat{\psi}(av)\right|^2 v^{-(2H_i+1)}dv \\ &= a^{2H_i+1}\sigma_i^2\, K_{H_i}(\psi), \end{aligned}$$

and the second part of the property is proved. ■

Proof. [**Proposition 3.1**] (we follow the same procedure that in Bardet, 2001). First, for $(a,a') \in]0,\infty[^2$, and $(b,b') \in \mathbb{R}^2$, using (43) we get :

$$\begin{aligned} \left|I\!E\left(d_{X_\rho}(a,b)d_{X_\rho}(a',b')\right)\right| &= \sqrt{aa'}\int_{\mathbb{R}} e^{-i(ab-a'b')\xi}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}d\xi \\ &\leq \frac{C}{(ab-a'b')^2}, \end{aligned} \tag{44}$$

when $|ab - a'b'| > 1$. The last inequality follows from assumption (A1) and a double integration by part (with $C > 0$). By this way, and with:

$$
\begin{aligned}
\mathrm{cov}\left(I_N(a)I_N(a')\right) &= \frac{1}{[N/a][N/a']}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\mathrm{cov}\left(d^2_{X_\rho}(a,p\Delta_N)d^2_{X_\rho}(a',p'\Delta_N)\right)\\
&= \frac{2}{[N/a][N/a']}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\mathrm{cov}^2\left(d_{X_\rho}(a,p\Delta_N)d_{X_\rho}(a',p'\Delta_N)\right),
\end{aligned}
$$

because $d_{X_\rho}(.,.)$ is a Gaussian random variable. Define $\lambda_N(a,a')$ the set of pair (p,p') such that $|ap\Delta_N - a'p'\Delta_N| < 1$, then for $(p,p') \in \lambda_N(a,a')$ we have

$$
\mathrm{cov}^2\left(d_{X_\rho}(a,p\Delta_N)d_{X_\rho}(a',p'\Delta_N)\right) \le I\!E d^2_{X_\rho}(a,.)I\!E d^2_{X_\rho}(a',.) \le M(a,a')
$$

with $M(a,a') > 0$. From the other hand, for a pair (p,p') satisfying $|ap\Delta_N - a'p'\Delta_N| \ge 1$, formula (44) induces that

$$
\mathrm{cov}^2\left(d_{X_\rho}(a,p\Delta_N)d_{X_\rho}(a',p'\Delta_N)\right) \le C^2|ap\Delta_N - a'p'\Delta_N|^{-4}.
$$

Therefore, we get

$$
\begin{aligned}
\mathrm{cov}\left(I_N(a)I_N(a')\right) &= \frac{2}{[N/a][N/a']}\left(\sum_{(p,p')\in\lambda_N(a,a')}\mathrm{cov}^2\left(d^2_{X_\rho}(a,p\Delta_N)d^2_{X_\rho}(a',p'\Delta_N)\right)+..\right.\\
&\qquad\left. ..+\sum_{(p,p')\notin\lambda_N(a,a')}\mathrm{cov}^2\left(d^2_{X_\rho}(a,p\Delta_N)d^2_{X_\rho}(a',p'\Delta_N)\right)\right)\\
&\le \frac{2}{[N/a][N/a']}\left(M\,|\lambda_N(a,a')|+\frac{C'}{\Delta_N^2}\iint_{|at-a't'|\ge 1}\frac{\mathbf{1}_{0\le\Delta_N t\le N/a}\mathbf{1}_{0\le\Delta_N t'\le N/a'}}{|at-a't'|^4}dtdt'\right)\\
&\le \frac{2}{[N/a][N/a']}\left(\frac{2MN}{aa'\Delta_N}+\frac{C'}{\Delta_N^2 a}\int_0^{\Delta_N N/a'}\int_1^N\frac{du}{u^4}dt'\right)\\
&\le \frac{M'}{N\Delta_N}+\frac{C''}{N\Delta_N}.
\end{aligned}
$$

This induces that $N\Delta_N\mathrm{cov}\left(I_N(a)I_N(a')\right) \to \gamma_\rho(a,a')$ when $N\to\infty$, with $\gamma_\rho(a,a') > 0$. More precisely,

$$
\begin{aligned}
\gamma_\rho(a,a') &= \lim_{N\to\infty}\frac{2aa'\Delta_N}{N}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\left(\sqrt{aa'}\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(ap-a'p')\Delta_N\xi}d\xi\right)^2\\
&= \lim_{N\to\infty}\frac{2a^2a'^2}{N\Delta_N}\int_{t=0}^{N\Delta_N/a}\int_{t'=0}^{N\Delta_N/a'}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(at-a't')\xi}d\xi\right)^2 dtdt'
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{2aa'}{N\Delta_N} \int_0^{N\Delta_N} \int_0^{N\Delta_N} \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2} e^{-i(t-t')\xi} d\xi \right)^2 dt dt' \\
&= \lim_{N\to\infty} \frac{2aa'}{N\Delta_N} \int_{-N\Delta_N}^{N\Delta_N} (N\Delta_N - |u|) \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du \\
&= \lim_{N\to\infty} \frac{2aa'}{N\Delta_N} \left[\int_{|u|\leq N\Delta_N} (N\Delta_N) \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 + \cdots \right. \\
&\qquad \left. \cdots + \int_{|u|>N\Delta_N} (N\Delta_N - |u|) \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du \right] \\
&= 2aa' \int_{I\!R} \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du + \lim_{N\to\infty} \frac{2aa'}{N\Delta_N} \mathcal{O}(1),
\end{aligned}
$$

from the previous bound of $\gamma_\rho(a,a') > 0$. As a consequence, we obtain the following central limit theorem :

$$
\sqrt{N\Delta_N} \left(I_N(a_j) - \int_{I\!R} \left| \frac{\overline{\widehat{\psi}}(a_j\xi)}{\rho(\xi)} \right|^2 d\xi \right)_{1\leq j\leq m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m(0,\Gamma),
$$

with $\Gamma = (\gamma_{k,l})_{1\leq k,l\leq m}$ and

$$
\gamma_{k,l} = 2a_k a_l \int_{I\!R} \left(\int_{I\!R} \frac{\overline{\widehat{\psi}}(a_k\xi)\widehat{\psi}(a_l\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du.
$$

Now, from the Slutsky theorem, Proposition 3.1 is proved. ■

Proof. [**Property 3.2**] First, we prove that for all $m \in I\!N$, and b_N such that $(T_N - b_N\Delta_N) \to \infty$ and $b_N\Delta_N \to \infty$ when $N \to \infty$,

$$
I\!E\left(d^2_{X_\rho}(a, b_N\Delta_N)\right) = I\!E\left(\frac{1}{\sqrt{a}} \int_0^{T_N} \psi(\frac{t}{a} - b_N\Delta_N) X_\rho(t) dt \right)^2 + \mathcal{O}\left(\frac{1}{T_N - b_N\Delta_N} + \frac{1}{b_N\Delta_N} \right). \tag{45}
$$

We have:

$$
\begin{aligned}
I\!E\left(d^2_{X_\rho}(a, b_N\Delta_N)\right) &= I\!E\left(\frac{1}{\sqrt{a}} \int_0^{T_N} \psi(\frac{t}{a} - b_N\Delta_N) X_\rho(t) dt \right)^2 + .. \\
&\quad .. \frac{2}{a} I\!E\left(\int_{I\!R} \int_{T_N - ab_N\Delta_N}^{\infty} \psi(\frac{u}{a})\psi(\frac{u'}{a}) X_\rho(u + ab_N\Delta_N) X_\rho(u' + ab_N\Delta_N) du du' \right) + .. \\
&\quad .. \frac{2}{a} I\!E\left(\int_{I\!R} \int_{-\infty}^{ab_N\Delta_N} \psi(\frac{u}{a})\psi(\frac{u'}{a}) X_\rho(u + ab_N\Delta_N) X_\rho(u' + ab_N\Delta_N) du du' \right);
\end{aligned}
$$

But for all $m \in \mathbb{N}$, for all $u \in \mathbb{R}$,

$$\int_{T_N - ab_N\Delta_N}^{\infty} \left| \psi(\frac{u'}{a}) \mathbb{E}\left(X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)\right)\right| du' \quad \leq$$
$$C^2|u+ab_N\Delta_N| \int_{T_N - ab_N\Delta_N}^{\infty} u'^{m+1} \left|\psi(\frac{u'}{a})\right| \frac{1}{u'^m} \frac{|u'+ab_N\Delta_N|}{u'} du,$$

because it exists $C>0$ such that $|\mathbb{E} X_\rho(t)X_\rho(t')| \leq C|tt'|$for $|t|>1$ and $|t'|>1$. Thus,

$$\mathbb{E}\left(\int_{\mathbb{R}}\int_{T_N - ab_N\Delta_N}^{\infty} \psi(\frac{u}{a})\psi(\frac{u'}{a})X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)dudu'\right) \quad \leq$$
$$\leq C' \frac{1}{(T_N - ab_N\Delta_N)^m}\int_{\mathbb{R}} \left|\psi(\frac{u}{a})\right| |u+ab_N\Delta_N| du$$
$$\leq C'' \frac{1}{(T_N - ab_N\Delta_N)^m} \qquad ,$$

when $T_N - ab_N\Delta_N \to \infty$, from the first condition of Assumption (A1), and with $C'>0$ and $C''>0$. The second part of the integral (between $-\infty$ and $-ab_N\Delta_N$) is obtained from the same trick. Thus, (45) is proved. We just have now to compared the integral and the sum. We use the relation (45) and:

$$\mathbb{E}\left(\frac{1}{\sqrt{a}}\int_0^{T_N}\psi(\frac{t}{a}-k\Delta_N)X_\rho(t)dt\right)^2 = \mathbb{E}\left(\frac{\Delta_N}{\sqrt{a}}\sum_{p=0}^{N}\psi(\frac{p\Delta_N}{a}-k\Delta_N)X_\rho(p\Delta_N)\right)^2 + \mathcal{O}(\Delta_N),$$

from a classical comparison of a Riemann sum and integral and because the function $\mathbb{E} X_\rho(t)X_\rho(t')$ is $\mathcal{C}^1$ on $\mathbb{R}^2$ except on the diagonal line $t=t'$. It finishes the proof of the first part of Property 3.2.

The second part of Property 3.2 is a consequence of the first part with a good choice of b_N and $m \in \mathbb{N}$. ∎

A.3 Proofs of section 4

To prove Proposition 4.1 and Proposition 4.2, we need bounds on the coefficients of the matrix Σ associated to the family of scales given in (28). This is stated in the following lemma.

Lemma A.1 *Let ψ verify Assumption (A1) with $\alpha > 1$, and Σ be the matrix defined by (16) with $a_k = f_k^{-1}$, where the frequencies $(f_k)_{k=0,\dots,M}$ are given by (28). Then we have* *i)* $s_{k,l} = 0$ *if* $|k - l| \geq \kappa$*;*

ii) If $\alpha > 1$*, then* $|s_{k,l}| \leq 2 f_0^{-2} \times L(\psi)$ *for all* $(k,l) \in \{1, \ldots, M\}^2$*, where* $L(\psi)$ *is given by (32).*

Proof. i) Let $k \geq l$ for instance, this implies $f_k \geq f_l$. But the support of $\xi \longmapsto \overline{\widehat{\psi}}(\xi f_k^{-1}) \widehat{\psi}(\xi f_l^{-1})$ is $[\alpha\, f_k, \beta f_k] \cap [\alpha\, f_l, \beta f_l]$ which is of measure 0 as soon as $\alpha\, f_k \geq \beta f_l$. Since this condition is equivalent to $k \geq l + \kappa$, we deduce that $s_{k,l} = 0$ in this case and by symmetry when $|k - l| \geq \kappa$.

ii) From (16), we have

$$
\begin{aligned}
|s_{k,l}| &\leq \frac{2}{f_k\, f_l\, \mathcal{I}_1(f_k^{-1})\, \mathcal{I}_1(f_l^{-1})} \times \int_{I\!R^+} \left| \frac{\overline{\widehat{\psi}}(u f_k^{-1})\, \widehat{\psi}(u f_l^{-1})}{\rho(u)^2} \right|^2 du \\
&\leq \frac{2}{f_k\, f_l\, \mathcal{I}_1(f_k^{-1})\, \mathcal{I}_1(f_l^{-1})} \times \int_{I\!R^+} \frac{\left|\widehat{\psi}(u f_k^{-1})\right|^2 \left|\widehat{\psi}(u f_l^{-1})\right|^2}{\rho(u)^4} du \\
&\leq \frac{2}{f_k\, f_l\, \mathcal{I}_1(f_k^{-1})\, \mathcal{I}_1(f_l^{-1})} \times \left(\int_{I\!R^+} \frac{\left|\widehat{\psi}(u f_k^{-1})\right|^4}{\rho(u)^4} du \right)^{1/2} \times \left(\int_{I\!R^+} \frac{\left|\widehat{\psi}(u f_l^{-1})\right|^4}{\rho(u)^4} du \right)^{1/2}
\end{aligned}
$$

where the two last lines follow from Plancherel formula and Cauchy-Schwartz inequality. Since (28) implies that $(f_k\, f_l)^{-1/2} \leq f_0^{-1}$, we get

$$
|s_{k,l}| \leq \frac{2}{f_0^2} \left(\frac{\mathcal{I}_2(f_k^{-1})}{\mathcal{I}_1(f_k^{-1})^2} \right)^{1/2} \times \left(\frac{\mathcal{I}_2(f_l^{-1})}{\mathcal{I}_1(f_l^{-1})^2} \right)^{1/2} \tag{46}
$$

where $\mathcal{I}_1(a)$ is given by (14) and $\mathcal{I}_2(f^{-1})$ is defined as follows

$$
\mathcal{I}_2(f^{-1}) = \int_{I\!R^+} \frac{\left|\widehat{\psi}(u)\right|^4}{\rho^4(u\, f)} du.
$$

We want to bound $\dfrac{\mathcal{I}_2(f_k^{-1})}{\mathcal{I}_1(f_k^{-1})^2}$ for every $k = 0, \ldots, M$. Assume first that $f_k \in [\dfrac{\omega_1}{\beta}, \dfrac{\omega_1}{\alpha}]$. As a consequence :

$$
\begin{aligned}
\mathcal{I}_1(f_k^{-1}) &= \int_\alpha^{\omega_1 f_k^{-1}} |\widehat{\psi}(u)|^2\, \rho(u f_k)\, du + \int_{\omega_1 f_k^{-1}}^\beta |\widehat{\psi}(u)|^2\, \rho(u f_k)\, du \\
&= f_k^{-(2H_0+1)}\, \sigma_0^2 \int_\alpha^{\omega_1 f_k^{-1}} |\widehat{\psi}(u)|^2\, u^{-(2H_0+1)}\, du + f_k^{-(2H_1+1)}\, \sigma_1^2 \int_{\omega_1 f_k^{-1}}^\beta |\widehat{\psi}(u)|^2\, u^{-(2H_1+1)}\, du.
\end{aligned}
$$

Since $\alpha > 1$, the map $H \mapsto u^{-(2H+1)}$ is decreasing for all $u \in [\alpha,\, \beta]$ which combined with $(H_0, H_1) \in [0,1]^2$ induces

$$
\mathcal{I}_1(f_k^{-1}) \geq \left[f_k^{-(2H_0+1)}\, \sigma_0^2 + f_k^{-(2H_1+1)}\, \sigma_1^2 \right] \times \int_\alpha^\beta |\widehat{\psi}(u)|^2\, u^{-3}\, du.
$$

Similarly, we get

$$\begin{aligned}
\mathcal{I}_2(f_k^{-1}) &= f_k^{-(4H_0+2)}\,\sigma_0^4 \int_\alpha^{\omega f_k^{-1}} |\widehat{\psi}(u)|^4\,(u)^{-(4H_0+2)}\,du + f_k^{-(4H_1+2)}\,\sigma_1^4 \int_{\omega f_k^{-1}}^{\beta} |\widehat{\psi}(u)|^4\,(u)^{-(4H_1+2)}\,du \\
&\leq \Big[f_k^{-(4H_0+2)}\,\sigma_0^4 + f_k^{-(4H_1+2)}\,\sigma_1^4\Big] \times \int_\alpha^\beta |\widehat{\psi}(u)|^2\,u^{-2}\,du.
\end{aligned}$$

We deduce

$$\frac{\mathcal{I}_2(f_k^{-1})}{\mathcal{I}_1(f_k^{-1})^2} \;\leq\; \frac{\Big(f_k^{-(4H_0+2)}\,\sigma_0^4 + f_k^{-(4H_1+2)}\,\sigma_1^4\Big)}{\Big(f_k^{-(2H_0+1)}\,\sigma_0^2 + f_k^{-(2H_1+1)}\,\sigma_1^2\Big)^2} \times L(\psi).$$

Since $(x^2+y^2) \leq (x+y)^2$ for all $(x,y) \in (I\!\!R^+)^2$, we deduce

$$\frac{\mathcal{I}_2(f_k^{-1})}{\mathcal{I}_1(f_k^{-1})^2} \;\leq\; L(\psi). \tag{47}$$

If $f_k \notin [\frac{\omega_1}{\beta}, \frac{\omega_1}{\alpha}]$, the proof is the same with $H_1 = H_0$ and $\sigma_1 = \sigma_0$. Thus the bound (47) is in force for every frequency f_k with $k = 0, \ldots, M$, combined with (46) this finishes the proof of the lemma. ■

Proof. [**Proposition 4.1**] (The calculations follow the the same lines as in [10]). Under the null hypothesis, we have a linear regression

$$\begin{aligned}
Y_\nu &= -(2H+1) \times \log(f_\nu) + \gamma + \varepsilon_\nu^{(N)} \qquad (48)\\
&= -\left[(2H+1)\log q\right] \times \nu + \gamma' + \varepsilon_\nu^{(N)} \qquad \text{for} \quad \nu = 0, \ldots, M,
\end{aligned}$$

from the definition (28) of f_ν and with

$$(N\Delta_N)^{1/2}\left(\varepsilon_\nu^{(N)}\right)_{0\leq\nu\leq M} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_{M+1}(0,\Sigma). \tag{49}$$

Since we have $(\Lambda+1)$ points in each box, following [10], we get for $\mu = \Lambda+\kappa, .., M-\Lambda$,

$$\begin{aligned}
D_1^{(N)}(\Lambda,\mu) &= \frac{6}{\Lambda(\Lambda+1)(\Lambda+2)\log q}\left\{\sum_{i=0}^{\Lambda}\left(i-\frac{\Lambda}{2}\right)\varepsilon_{\mu-\Lambda-\kappa+i}^{(N)} - \sum_{i=0}^{\Lambda}\left(i-\frac{\Lambda}{2}\right)\varepsilon_{\mu+i}^{(N)}\right\}, \\
&= \Gamma(\mu).(\varepsilon_i^{(N)})_{0\leq i\leq M}.
\end{aligned}$$

¿From (49), we deduce for $\mu = \Lambda+\kappa, .., M-\Lambda$,

$$(N\Delta_N)^{1/2}\,\Gamma(\mu).(\varepsilon_i^{(N)})_{0\leq i\leq M} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0,\Gamma(\mu)\,\Sigma\,\Gamma(\mu)'). \tag{50}$$

But,

$$\begin{aligned}
\mathbb{P}_{(H_0)}\left(\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda}|D_1^{(N)}(\Lambda,\mu)|>\eta\right) &\leq \sum_{\mu=\Lambda+\kappa}^{M-\Lambda}\mathbb{P}_{(H_0)}\left(|D_1^{(N)}(\Lambda,\mu)|>\eta\right)\\
&\leq \sum_{\mu=\Lambda+\kappa}^{M-\Lambda}\frac{\mathbb{E}\left\{\left(D_1^{(N)}(\Lambda,\mu)\right)^2\right\}}{\eta^2}\\
&\leq \left(\frac{M-2\Lambda-\kappa+1}{\eta^2}\right)\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda}\left\{\mathrm{var}\left(D_1^{(N)}(\Lambda,\mu)\right)\right\}.
\end{aligned}$$

We know that for every $\mu\in\{\Lambda+\kappa,\cdots,M-\Lambda\}$, $\mathbb{E}\left\{\left(D_1^{(N)}(\Lambda,\mu)\right)^3\right\}$ exists (because $D_1^{(N)}(\Lambda,\mu)$ is a linear combination of logarithms of sum of squared Gaussian random variables). Then, from (50), we obtain for every $\mu\in\{\Lambda+\kappa,\cdots,M-\Lambda\}$, and for N large enough :

$$\mathrm{var}\left(D_1^{(N)}(\Lambda,\mu)\right)\sim\frac{\Gamma(\mu)\,\Sigma\,\Gamma(\mu)'}{N\Delta_N}.$$

But, from Lemma A.1, we have for $\mu=\Lambda+\kappa,\cdots,M-\Lambda$,

$$\begin{aligned}
\Gamma(\mu)\,\Sigma\,\Gamma(\mu)' &= \left(\frac{6}{\Lambda(\Lambda+1)(\Lambda+2)\log q}\right)^2\\
&\quad\times\left[\sum_{i=1}^{\Lambda}\sum_{j=1}^{\Lambda}\left(i-\frac{\Lambda}{2}\right)\left(j-\frac{\Lambda}{2}\right)\left(s_{\mu+i,\mu+j}+s_{\mu-\Lambda-\kappa+i,\mu-\Lambda-\kappa+j}\right)\right]\\
&\leq \left(\frac{6}{\Lambda(\Lambda+1)(\Lambda+2)\log q}\right)^2\times\left(\frac{\Lambda}{2}\right)^2(4\kappa-2)(\Lambda+1)\max_{k,l=1,\ldots,\Lambda}|s_{k,l}|\\
&\leq \frac{36\,(2\kappa-1)}{f_0^2\,(\Lambda+1)(\Lambda+2)^2\log^2 q}\;L(\psi).
\end{aligned}$$

Then, we deduce that for N large enough :

$$N\,\Delta_N\times\mathrm{var}\left(D_1^{(N)}(\Lambda,\mu)\right)\leq 72\frac{\kappa}{f_0^2\,\Lambda^3\,\log^2 q}\;L(\psi).$$

Thus, using (25), we get for N large enough : $\mathrm{var}\left(D_1^{(N)}(\Lambda,\mu)\right)\leq\dfrac{C_1}{N\Delta_N}$. With (28), this induces (31) and finishes the proof of the proposition. ∎

Proof. [**Proposition 4.2**] The proof is divided into two steps : first, we suppose that the frequency change ω_1 corresponds to one of the chosen frequencies. In the second step, we consider the general case. To simplify the notations, we will denote into the rest of the proof

$$\beta_N=\mathbb{P}_{(H1)}\left(\max_{\Lambda+\kappa\leq\mu\leq M-\Lambda}|D_1^{(N)}(\Lambda,\mu)|<\eta\right).$$

First step. We suppose that there exists $\nu_0 \in \{\Lambda+\kappa, \dots, M-\Lambda\}$ such that $f_{\nu_0} = \dfrac{\omega_1}{\alpha}$. This induces

$$\beta_N \;\leq\; \mathbb{P}_{(H1)}\left(|D_1^{(N)}(\Lambda,\, \nu_0)| < \eta\right).$$

We have a linear regression with the parameters (H_1, σ_1), resp. (H_0, σ_0), when $\nu \geq \nu_0$, resp. $\nu + \kappa \leq \nu_0$, since $\log \beta/\alpha = \kappa \log q$; with the notations of the previous proof, using (23, 24, 25), we deduce

$$D_1^{(N)}(\Lambda,\, \nu_0) = -(H_1 - H_0) + \Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M}.$$

First consider the case $(H_1 - H_0) \geq d_0$. In this case,

$$|D_1^{(N)}(\Lambda,\, \nu_0)| < \eta \quad \Rightarrow \quad \Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M} > d_0 - \eta,$$

which leads to

$$\begin{aligned}\beta_N \;&\leq\; \mathbb{P}_{(H1)}\left(\Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M} \,>\, d_0 - \eta\right)\\ &\leq\; \frac{1}{(d_0-\eta)^2}\, \mathbb{E}\left|\Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M}\right|^2 \leq \frac{1}{N\Delta_N} \times \frac{C_1}{(d_0-\eta)^2} \end{aligned} \tag{51}$$

for N large enough, from the previous proof. In the case $(H_1 - H_0) < d_0$, we have $D_1^{(N)}(\Lambda,\, \nu_0) = -(H_1 - H_0) + \Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M}$, which leads to

$$\beta_N \;\leq\; \mathbb{P}_{(H1)}\left(-\Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\leq i\leq M} \,>\, d_0 - \eta\right)$$

and after to (51).

Second step. We just give the proof in the case $(H_1 - H_0) > d_0$. By symmetry, the same kind of proof works when $(H_1 - H_0) < d_0$. We suppose that for every $\nu \in \{0, 1, \cdots, M\}$, $f_\nu \neq \dfrac{\omega_1}{\alpha}$. However, there exists $\nu_0 \in \{\Lambda+\kappa, \dots, M-\Lambda\}$, such that

$$f_{\nu_0} < \omega_1/\alpha \,<\, f_{\nu_0+1}. \tag{52}$$

In this case, we have a linear regression with parameters (H_1, σ_1), resp. (H_0, σ_0), when $\nu \geq \nu_0 + 1$, resp. $\nu + \kappa \leq \nu_0$, but at the frequency $f = \omega_1 \alpha^{-1}$, resp. $f = \omega_1 \beta^{-1}$, we still have a linear regression with parameters (H_1, σ_1), resp. (H_0, σ_0). The aim of the following calculations is to control the resulting difference. Thus we want to bound the test statistic

$D_1^{(N)}(\Lambda,\,\mu)$ at the frequencies $\mu=\nu_0$ or $\mu=\nu_0+1$. From (29, 21), we deduce

$$\begin{aligned} D_1^{(N)}(\Lambda,\,\nu_0) &= -(H_1-H_0)+\Gamma(\nu_0).(\varepsilon_i)_{0\le i\le M} \\ &- \frac{3}{(\Lambda+1)\,(\Lambda+2)\,\log(q)}\times\left[\log\mathcal{I}_1(f_{\nu_0}^{-1})-\log\mathcal{I}_1(\alpha\omega_1^{-1})\right], \end{aligned}$$

and

$$\begin{aligned} D_1^{(N)}(\Lambda,\,\nu_0+1) &= -(H_1-H_0)+\Gamma(\nu_0+1).(\varepsilon_i)_{0\le i\le M} \\ &- \frac{3}{(\Lambda+1)\,(\Lambda+2)\,\log(q)}\times\left[\log\mathcal{I}_1(f_{\nu_0+1-\kappa}^{-1})-\log\mathcal{I}_1\left(\beta\omega_1^{-1}\right)\right]. \end{aligned}$$

But, from (14), we get

$$\mathcal{I}_1(f^{-1})=2\sigma_0^2\int_\alpha^{f^{-1}\omega_1}|\widehat{\psi}(u)|^2\,(fu)^{-(2H_0+1)}\,du+2\sigma_1^2\int_{f^{-1}\omega_1}^{\beta}|\widehat{\psi}(u)|^2\,(fu)^{-(2H_1+1)}\,du, \quad (53)$$

and

$$\begin{aligned} \log\mathcal{I}_1(\alpha\omega_1^{-1}) &= -(2H_1+1)\log\left(\omega_1\alpha^{-1}\right)+\log\left(\sigma_1^2\right)+\log\left(K_{H_1}(\psi)\right), &(54)\\ \log\mathcal{I}_1\left(\beta\omega_1^{-1}\right) &= -(2H_0+1)\log\left(\omega_1\beta^{-1}\right)+\log\left(\sigma_0^2\right)+\log\left(K_{H_0}(\psi)\right). &(55) \end{aligned}$$

Formula (54), resp. (55), follows from $fu\ge\omega_1$ for all $u\in[\alpha,\,\beta]$ as soon as $f\ge\omega_1\alpha^{-1}$, resp. $fu\le\omega_1$ for all $u\in[\alpha,\,\beta]$ as soon as $f\le\omega_1\beta^{-1}$. We should consider two different cases to obtain lower bounds for $D_1^{(N)}(\Lambda,\,\nu_0)$ and $D_1^{(N)}(\Lambda,\,\nu_0+1)$.
i) if $f_{\nu_0}\,\alpha\ge1$, then $v=f_{\nu_0}u\ge1$ for all u in $[\alpha,\,f_{\nu_0}^{-1}\omega_1]$. Therefore the map $H\mapsto v^{-(2H+1)}$ is decreasing when $v\ge1$ and $H>0$. Using (53) and $H_1>H_0$, we deduce that

$$\begin{aligned} \mathcal{I}_1(f_{\nu_0}^{-1}) &\ge 2\left(\sigma_0^2\wedge\sigma_1^2\right)\times\int_\alpha^\beta|\widehat{\psi}(u)|^2\,(f_{\nu_0}u)^{-(2\,H_1+1)}\,du \\ &ge \left(\sigma_0^2\wedge\sigma_1^2\right)\times K_{H_1}(\psi)\times f_{\nu_0}^{-(2H_1+1)}, \end{aligned}$$

where $(a\wedge b)$ denote the minimum of the two real numbers a and b. Combined with (54) and using $H_1>0$ and $\log(\omega_1\alpha^{-1})>\log(f_{\nu_0})$, we get

$$\begin{aligned} \log\mathcal{I}_1(f_{\nu_0}^{-1})-\log\mathcal{I}_1(\alpha\omega_1^{-1}) &\ge \log\left(\sigma_0^2\wedge\sigma_1^2\right)-\log\sigma_1^2-(2H_1+1)\left[\log(f_{\nu_0})-\log(\omega_1\alpha^{-1})\right] \\ &\ge -2\left|\log\left(\sigma_1/\sigma_0\right)\right|. \end{aligned}$$

This induces the following bounds

$$\begin{aligned} \eta &\ge -D_1^{(N)}(\Lambda,\,\nu_0) \\ &\ge (H_1-H_0)-\Gamma(\nu_0).(\varepsilon_i^{(N)})_{0\le i\le M}-\frac{6}{(\Lambda+1)\,(\Lambda+2)\,\log(q)}\,\left|\log\left(\sigma_1/\sigma_0\right)\right|. \end{aligned}$$

Finally, we have proved that

$$\text{if} \quad \log f_{\nu_0} + \log \alpha \geq 0, \quad \max_{\Lambda+\kappa \leq \mu \leq M-\Lambda} |D_1^{(N)}(\Lambda,\mu)| < \eta \quad \text{and} \quad H_1 - H_0 \geq d_0 \quad (56)$$

$$\text{then} \quad \Gamma(\nu_0).(\varepsilon_i^{(N)})_{0 \leq i \leq M} \geq d_0 - \eta - \frac{6}{(\Lambda+1)\,(\Lambda+2)\,\log(q)}\,|\log(\sigma_1/\sigma_0)|\,.$$

ii) if $f_{\nu_0+1-\kappa}\beta \leq 1$, then $v = f_{\nu_0+1-\kappa}u \leq 1$ for all u in $[f^{-1}_{\nu_0+1-\kappa}\omega_1, \beta]$. Therefore the map $H \mapsto v^{-(2H+1)}$ is increasing and we get

$$\mathcal{I}_1(f^{-1}_{\nu_0+1-\kappa}) \;\geq\; \left(\sigma_0^2 \wedge \sigma_1^2\right) \times K_{H_0}(\psi) \times f^{-(2H_0+1)}_{\nu_0+1-\kappa},$$

which implies

$$\log \mathcal{I}_1(f^{-1}_{\nu_0+1-\kappa}) - \log \mathcal{I}_1(\beta\omega_1^{-1}) \geq -2\,|\log(\sigma_1/\sigma_0)| - (2H_0+1)\,\left[\log(f_{\nu_0+1-\kappa}) - \log(\omega_1\beta^{-1})\right].$$

Straightforward calculations shows that the condition $f_{\nu_0+1-\kappa}\beta \leq 1$ is equivalent to $\log f_{\nu_0} + \log\alpha \leq -\log q$ and thus $\log(f_{\nu_0+1-\kappa}) - \log(\omega_1\beta^{-1}) = \log(f_{\nu_0+1}) - \log(\omega_1\alpha^{-1}) \in [0, \log q]$. This induces that

$$\log \mathcal{I}_1(f^{-1}_{\nu_0+1-\kappa}) - \log \mathcal{I}_1(\beta\omega_1^{-1}) \;\geq\; -2\,|\log(\sigma_1/\sigma_0)| - 3\log q$$

and after that

$$\text{if} \quad \log f_{\nu_0} + \log \alpha \leq -\log q, \quad \max_{\Lambda+\kappa \leq \mu \leq M-\Lambda} |D_1^{(N)}(\Lambda,\mu)| < \eta \quad \text{and} \quad H_1 - H_0 \geq d_0 \quad (57)$$

$$\text{then} \quad \Gamma(\nu_0+1).(\varepsilon_i^{(N)})_{0 \leq i \leq M} \geq d_0 - \eta - \frac{3}{(\Lambda+1)\,(\Lambda+2)}\left(3 + \frac{2\,|\log(\sigma_1/\sigma_0)|}{\log(q)}\right).$$

Since the log-frequencies $\log f_\nu$ are discrete with a lag $\log q$, we have a dichotomy and rather assumption (56), rather (57) is satisfied. In both case, the same arguments than in the first step induce that for N large enough

$$\beta_N \;\leq\; \frac{1}{N\Delta_N} \times \frac{C_1}{(d_0 - \widetilde{\eta})^2}$$

which finishes the proof. ■

Références

[1] Abry, P., Flandrin, P., Taqqu, M. and Veitch, D. (2001). Self-similarity and long-range dependence through the wavelet lens. To appear in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser.

[2] Bardet J.M. (2000a). Testing for the presence of self-similarity of Gaussian time series having stationary increments. *J. of Time Series Anal.* 21, p. 497-516.

[3] Bardet J.M. (2000b). Les cours d'actifs financiers sont-ils autosimilaires? *Journal de la SFdS* 141, p. 137-148.

[4] Bardet J.M. (2001). A statistical study of the wavelet analysis of fractional Brownian motion. To appear in IEEE Trans. on Inform. Theory.

[5] Bardet J.M., Lang G., Moulines E. and Soulier P. (2000). Wavelet estimator of long-range dependent processes. *Statistical Inference for Stochastic Processes* 3, p. 85-99.

[6] Benassi, A. and Deguy, S. (1999). Multi-scale fractional Brownian motion : definition and identification. Preprint LAIC.

[7] Benassi, A., Jaffard, S. and Roux, D. (1997). Elliptic Gaussian random processes. *Rev. Mathemàtica Iberoamericana* 13, p. 19-90.

[8] Bertrand, P. (2000). A local method for estimating change points : the hat-function. *Statistics* 34, p. 215-235.

[9] Bertrand, P., Bardet, J.M, Dabonneville, M., Mouzat, A. and Vaslin, P. (2001). "Automatic determination of the different control mechanisms in upright position by a wavelet method", to appear in *I.E.E.E. Engineering in Medicine and Biology Society.*

[10] Bertrand, P. and Guillin, A. Multiple changepoints detection in linear regression. In preparation.

[11] Cheridito, P. (2000). Arbitrage in Fractional Brownian Motion Models. Preprint.

[12] Cohen, S. (2000). Champs localement auto-similaires. In *Fractals et loi d'échelles*, P.Abry, P. Goncalvès and J. Lévy-Vehel editors, Hermès, Paris

[13] Collins J.J., De Luca C.J. (1993). Open-loop and closed-loop control of posture : A random walk analysis of center-of-pressure trajectories. *Experimental Brain Research* 9, p. 308-318.

[14] Cramér, H.; Leadbetter, M. R. (1967), *Stationary and related stochastic processes. Sample function properties and their applications.* Wiley and Sons.

[15] Dahlhaus R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* 17, p. 1749-1766

[16] Flandrin, P. (1992). Wavelet analysis and synthesis of fractional Brownian motion. *IEEE Trans. on Inform. Theory* 38, p. 910-917.

[17] Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent Gaussian time series. *Ann. Statist.* 14, p. 517-532.

[18] Giraïtis L. and Surgaïlis D. (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its applications to the asymptotic normality of Whittle estimate. *Prob. Th. and Rel. Fields.* 86, p. 87-104.

[19] Hunt, G.A. (1951). Random Fourier transforms. *Trans. of the American Society of Civil Engineers* 116, p. 770-808.

[20] Mandelbrot, B., and Van Ness J. (1968). Fractional Brownian motion, fractional noises and applications. *SIAM review* 10, p.422-437.

[21] Meyer, Y. (1990), *Ondelettes et opérateurs*, Hermann.

[22] Perrin, E., Harba, R., Berzin-Joseph, C., Iribarren, I. and Bonami, A. (2001). Nth-order fractional Brownian motion and fractional Gaussian noises. To appear in *IEEE Trans. on Signal Proc.*

[23] Revuz,A. and Yor, M., (1991)*Continuous Martingales and Brownian Motion*, 3rd ed. Springer Verlag, New-York.

[24] Rogers, L.C.G. (1997). Arbitrage with fractional Brownian motion. *Mathematical Finance* 7, p. 95-105.

[25] Samorodnitsky, G. and Taqqu M. S. (1994), *Stable non-Gaussian Random Processes*, Chapman & Hall.

[26] Yaglom, A.M. (1958). Correlation theory of processes with stationary random increments of order n. *Trans. A.M.S.* 8, p.87-141.