# Definition, properties and wavelet analysis of multiscale fractional Brownian motion

April 16, 2003

JEAN-MARC BARDET* and PIERRE BERTRAND**

* (Corresponding Author) *Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 31062 Toulouse, France. Tel : 33-0561558573, Fax : 33-0561556089. E-mail: bardet@cict.fr*

** *Université Blaise Pascal (Clermont-Ferrand II), 34 avenue Carnot, 63000 Clermont-Ferrand, France. E-mail: Pierre.Bertrand@math.univ-bpclermont.fr*

In some applications, for instance finance, biomechanics, turbulence or internet traffic, it is relevant to model data with a generalization of a fractional Brownian motion for which the Hurst parameter $H$ is depending on the frequency. In this contribution, we describe the multiscale fractional Brownian motions which present a parameter $H$ as a piece-wise constant function of the frequency. We provide the main properties of these processes and propose a statistical method based on wavelet analysis to estimate the different parameters.

*Keywords:* Fractional Brownian motion; Long-range dependence; Path regularity; Self-similarity; Wavelet analysis.

# 1 Introduction

Fractional Brownian motions (f.B.m.) were popularized by Mandelbrot and Van Ness (1968) who suggest the study of their properties as a typical example of non-Markovian process. F.B.m. are Gaussian processes with stationary increments with a given covariance structure. Formally, a fractional Brownian motion $B_H = \{B_H(t),\, t \in I\!\!R_+\}$ could be defined as a real centered Gaussian process with stationary increments such that $B_H(0) = 0$ and $I\!\!E\,|B_H(s) - B_H(t)|^2 = \sigma^2\,|t-s|^{2H}$, $\forall (s,t) \in I\!\!R_+ \times I\!\!R_+$ where $H \in ]0,1[$ and $\sigma > 0$. This process is characterized by two parameters : the Hurst index $H$ and the scale parameter $\sigma$. The Hurst index $H$ is linked to different properties of the f.B.m. :

1. Self-similarity of the process, *i.e.* for all $\lambda \in I\!\!R^+$,

$$(B_H(\lambda t))_{t\in I\!\!R_+} \overset{(d)}{=} \left(\lambda^H\, B_H(t)\right)_{t\in I\!\!R_+}.$$

Following Mandelbrot and Van Ness (1968) [18], *" the concept of self-similarity is a form of invariance with respect to changes of time scale "*. For this reason, the same parameter $H$ drives the two following properties of the f.B.m. :

2. High frequency behavior : the regularity of the paths. $B_H$ is Hölder continuous of order $\alpha$, for all positive real number $\alpha < H$;

3. Low frequency behavior : short or long memory property of the increments. Let $Y(t) = B_H(t+1) - B_H(t)$, for $t \in I\!\!R_+$, $Y$ is a centered stationary Gaussian process corresponding to the increment of the process $B_H$. A stationary $I\!\!L^2$-process $Y = \{Y(t), t \in I\!\!N\}$ is said to be a long-range dependent process (or long-memory process) when

$$\sum_{k=-\infty}^{\infty} |r(k)| = \infty,$$

where $r(k) = \mathrm{cov}\,(Y(k), Y(0))$ is the covariogram of $Y$. For the f.B.m. we have $|r(k)| \sim \sigma^2\, H(2H-1)\, k^{2H-2}$ when $|k| \to +\infty$. Thus, when $H > 1/2$, the increments of a f.B.m. are long-memory process and when $H \leq 1/2$, a short-memory process.

The invariance with respect to changes of time scale links the high frequency behavior and the low frequency behavior. However, in numerous applications, this link can appear as too strict and artificial. For instance, statistical studies on financial data have shown some long memory properties of the price process, see for e.g. Willinger *et al.* [22]. From the other hand, economic considerations say it should not be possible to make profit without any risk and this implies that the price process should be a semi-martingale but the f.B.m. is not a semi-martingale except when $H = 1/2$. Anyway this statement would be more precise if we say that the Hurst index should be $H = 1/2$ at high frequencies. In this case the contradiction between the empirical evidence of long memory and the theoretical reasoning follows from the invariance with respect to changes of time scale and would be solved if we put an end to this invariance principle. In turbulence phenomena, the assumption of scale-invariance is assume within the inertial range (see [16, p.104] or [19]). Fundamental and experimental physic show that the scale invariance is valid for a large range of scales excluding both the microscopically and the astronomically ones. In biomechanics, Collins and de Luca (1993) introduced a fractional process with two different Hurst indexes at low and high frequencies, for modeling the trajectories of the center of pressure of human been in upright position.

The previous examples exhibit the interest of the modeling by a Gaussian processes having stationary increments with a Hurst index varying as a piecewise function of the frequencies. To our knowledge, these kinds of processes have been introduced (implicitly) by Collins and de Luca (1993), by Rogers (1997) in a discussion where he rejects the f.B.m. (with a constant Hurst index) as an admissible model for the stock prices, and by Benassi and Deguy (1999) for image analysis and image synthesis.

The aim of this paper is, firstly, the establishment of the definition and main properties of such a process, so-called multiscale fractional Brownian motion, (see Section 2) and, secondly, its analysis by wavelet as a relevant way of its statistical study (see Section 3). Indeed, the wavelet analysis is built from scale changes and is thus well adapted to study a stochastic process having different behavior following frequencies. More precisely, the empirical variance of wavelet coefficients provides an interesting semi-parametric method for estimating the different parameters of the process. In the appendices A1 and A2, we develop the examples of applications in finance and biomecanic. The proofs are given in the appendices B1 and B2.

# 2 Description of the model

The fractional Brownian motion (f.B.m.) has been introduced by Kolmogorov (1940). Afterward, Mandelbrot (1965) first noticed its relevance for modeling natural phenomenon (for instance the level of the Nil river). Since the seminal paper Mandelbrot and Van Ness (1968) [18], which investigates its basic properties, many other applications has been proposed. However, the f.B.m. appears as a too simple model for correctly modelling numerous natural phenomena. This suggest the introduction of some generalizations. The f.B.m. has different representations which could also be used as definition : the moving average representations, the harmonizable representation. We propose Samorodnitsky and Taqqu (1994), [21, chap. 7] as a reference book on the matter. The harmonizable representation of the f.B.m. ([21, (7.2.12), p.328] is well adapted to our problem and is defined by

$$B_H(t) = \int_{I\!\!R} \frac{\left(e^{it\xi} - 1\right)}{|\xi|^{H+1/2}} \, \widehat{W}(d\xi) \tag{1}$$

where $W(dx)$ is a standard Brownian measure and $\widehat{W}(d\xi)$ its Fourier transform. A natural generalization consists to replace in (1) the Hurst index by a piecewise function depending on the frequency. We call it the multiscale fractional Brownian motion and we define it as following :

**Definition 2.1** *For $K \in \mathbb{N}$, a $(M_K)$-multiscale fractional Brownian motion (simplify by $(M_K)$-f.B.m.) $X = \{X(t), t \in \mathbb{R}_+\}$ is a process such that*

$$X(t) = \int_{\mathbb{R}} \frac{(e^{it\xi} - 1\ )}{\rho(\xi)}\, \widehat{W}(d\xi) \quad \text{for } t \in \mathbb{R}_+ \tag{2}$$

*with*

- *$W(d\xi)$ is a standard Brownian measure and $\widehat{W}(d\xi)$ its Fourier transform in the distribution meaning.*
- *for $i = 0, \ldots, K$, there exist $(\omega_i, \sigma_i, H_i) \in (\mathbb{R}_+, \mathbb{R}_+, ]0,1[)$ such that $\rho(\xi) = \sigma_i^{-1}|\xi|^{H_i+1/2}$ for $|\xi| \in [\omega_i, \omega_{i+1}[$ with $\omega_0 = 0 < \omega_1 < \ldots < \omega_K < \omega_{K+1} = +\infty$ by convention.*

The $(M_K)$-f.B.m. was first introduced in Benassi and Deguy (1999) in the case of only one change. Until the end of this article, we restrict ourselves to one dimensional process depending on a parameter $t \in \mathbb{R}_+$, but a generalization to random fields could also be done. The proof of the existence of a continuous version of a $(M_K)$-f.B.m. could be found in Cramér and Leadbetter (1967). A $(M_K)$-f.B.m. presents $K$ spectral changes. A $(M_0)$-f.B.m. is a fractional Brownian motion with a Hurst parameter $H_0$, and in its harmonizable representation, a $(M_K)$-f.B.m. has the same behavior than a fractional Brownian motion with a Hurst parameter $H_i$ for frequencies $|\xi| \in [\omega_i, \omega_{i+1}[$. Therefore we could expect that a $(M_K)$-f.B.m. path would have the regularity $H_K$ (at high frequencies) and a long term dependence driven by $H_0$ (at low frequencies). The first point is verified, but Property 2.2 shows that the second one is a little bit more complicated.

**Property 2.1** *Let $X$ be a $(M_K)$-f.B.m.*

1. *$X$ is a Gaussian centered process with stationary increments.*
2. *For a time step $\delta > 0$, let the variogram of $X$ be*

$$\mathcal{V}(\delta) := \mathbb{E}\left(X(t+\delta) - X(t)\right)^2 \;=\; 4\sum_{i=0}^{K} \delta^{2H_i}\, \sigma_i^2 \int_{\delta\omega_i}^{\delta\omega_{i+1}} \frac{(1-\cos v)}{v^{2H_i+1}} dv. \tag{3}$$

3. *We have the decomposition*

$$X(t) \quad = \quad B_{H_K}(t) + R(t), \tag{4}$$

*where $B_{H_K}$ is a f.B.m. with Hurst index $H_K$ and $R$ a continuous process with finite variation. Thus $X$ has the same regularity than the f.B.m. $B_{H_K}$. Particularly, the trajectories of $X$ are a.s. of Hölder regularity $\alpha$, for every $\alpha \in [0,\, H_K[$.*

Therefore a $(M_K)$-f.B.m. shares two properties with a f.B.m. of parameter $H_K$ : they are continuous Gaussian centered processes with stationary increments and $H_K$-Hölderian regularity. However, Formula (3) shows a substancial difference between both the processes : for all $\delta \in I\!R_+$, the variance of a $(M_K)$-f.B.m. in $\delta$ is a function depending on $\delta$ and all the parameters $(H_i, \omega_i, \sigma_i)_{0\leq i\leq K}$. As a consequence, the variance of a $(M_K)$-f.B.m. is not a self-similar function and the famous log-variogram method is not relevant for the estimation of the different parameters.

Now, in view of the study of the low frequency behavior, we consider process of the increments of $X$, $Y = \{Y(t), t \in I\!R_+\}$ of $X$, with

$$Y(t) = X(t+1) - X(t), \quad \text{for } t \in I\!R_+.$$

$Y$ is a centered stationary Gaussian process. Moreover, we have the following property :

**Property 2.2** *Let $X$ be a $(M_K)$-f.B.m. with $K \geq 1$ and $Y$ its increments defined as previously. Define $r(n) = cov\,(Y(n), Y(0))$, the correlogram of the increments of $X$. Then*

$$\text{if } H_0 > \frac{1}{2}, \quad r(n) = \left(\frac{\pi\sigma_0^2(2H_0-1)}{\Gamma(2H_0)\sin(\pi H_0)}\right)\frac{1}{n^{2-2H_0}} + \mathcal{O}\left(\frac{1}{n}\right), \tag{5}$$

$$\text{if } H_0 \leq \frac{1}{2}, \quad r(n) = \frac{P(n)}{n} + \mathcal{O}\left(\frac{1}{n^{2-2H_0}}\right), \tag{6}$$

*with $P(n)$ a trigonometric polynomial depending only on $(H_i, \omega_i, \sigma_i)_{0\leq i\leq K}$ such that*

$$P(n) \quad = \quad 8\sum_{i=1}^{K}\sin^2\left(\frac{\omega_i}{2}\right) \times \left(\frac{\sigma_{i-1}^2}{\omega_{i-1}^{2H_i-1}} - \frac{\sigma_i^2}{\omega_i^{2H_i}}\right) \times \sin(n\omega_i). \tag{7}$$

¿From this property we derive the asymptotic behavior of the dependence of the increments of $X$. Then, the long-range property of the increments of the process $Y$ depends on $H_0$, but not only on this value. More precisely, if $H_0 > 1/2$, the long-range (or low frequencies) behavior of $Y$ is the same that the behavior of a fractional Gaussian noise with parameter $H_0$. Thus, $Y$ is a long-range dependent process. If $H_0 \leq 1/2$, the process $Y$ is a long-range dependent process except when $(H_i, \omega_i, \sigma_i)_{0 \leq i \leq K}$ verifies a relation such that the trigonometric polynomial $P(n)$ of (7) is vanished.

**Remark 2.1** *If we consider the spectral density function of the stationary process $Y$ in the meaning of Cramér and Leadbetter (1967, p. 136), i.e. the function $f$ such that* $r(t) = \int_0^\infty \cos(\lambda t) f(t) dt$, *we have* $f(t) = \dfrac{\sin^2(t/2)}{\rho^2(t)}$. *As a consequence, $f$ is not a continuous function except if for all $i = 1, \cdots, K$, $f(\omega_i) = \lim_{t \to \omega_i^-} f(t)$, i.e. the same condition that $P(n) = 0$ for all $n \in \mathbb{N}$. Since $f$ is not a continuous function then $\sum_{k \in \mathbb{N}} |r(k)| = \infty$ and thus $Y$ is a long-range dependent process. For instance, if $K = 1$ and $\dfrac{\sigma_0^2}{\omega_1^{2H_0+1}} \neq \dfrac{\sigma_1^2}{\omega_1^{2H_1+1}}$, $f$ is discontinuous in $\omega_1$ and $Y$ is a long-range dependent process even if $H_0 < 1/2$ and $H_1 < 1/2$.*

# 3 A statistical study based on wavelet analysis

Let $X$ be a $(M_K)$-f.B.m. defined by (2). We observe one path of the process $X$ on the interval $[0, T_N]$ at the discrete times $t_i = i \times \Delta_N$ for $i = 1, \ldots, N$ and $T_N = N \times \Delta_N$. We want to estimate the parameters of the $(M_K)$-f.B.m. that is $(H_0, H_1, \ldots, H_K)$, $(\sigma_0, \sigma_1, \ldots, \sigma_K)$ and $(\omega_1, \ldots, \omega_K)$. We consider the asymptotic $N \to +\infty$, $\Delta_N \to 0$ and $T_N \to +\infty$.

Even if the model is defined here as a parametric model, we prefer to use a semi-parametric statistics for different reasons. Firstly, the covariance of the increments has the asymptotic behavior described in Property 2.2 and if $H_0 < 1/2$, this behavior cannot be written as $r(n) = L(n)n^{-d}$ with $L$ a slow varying function. Then, the classical results (see Fox and Taqqu, 1986, Dahlhaus, 1989 or Giraitis and Surgailis, 1990) on the asymptotic behaviors of parametric (maximum likelihood estimator and Whittle maximum likelihood) estimators cannot be used. As a consequence, a new and difficult study of the convergence rate of such estimators should be done, without the guarantee to obtain good properties (another difficulty is implied by the discontinuity of the spectral density function as seen in Remark 2.1).

Secondly, the following semi-parametric statistics is more robust than a parametric one if the model is misspecified. Consider the example where the function $H(\xi)$ is a not exactly a piece-wise constant function, but a constant function on several intervals (in other intervals, $H(\xi)$ is some unknown function). In such a case, we cannot expect that a parametric estimator could be useful while the following semi-parametric method will remain relevant.

Therefore, we prefer here to use a method based on a wavelet analysis. This method has been introduced by Flandrin (1992) and was developed by Abry *et al.* (2001) and Bardet *et al.* (2000). Let $\psi$ be a "mother" wavelet such that:

**Assumption (A1):** $\psi : \mathbb{R} \mapsto \mathbb{R}$ *is a* $\mathcal{C}^\infty$ *function verifying:*

- *for all* $m \in \mathbb{R}$, $\int_{\mathbb{R}} |t^m \psi(t)|\, dt < \infty$,
- *its Fourier transform* $\widehat{\psi}(\xi)$ *is an even function compactly supported on* $[-\beta, -\alpha] \cup [\alpha, \beta]$ *with* $0 < \alpha < \beta$.

These conditions are sufficiently mild and are satisfied by famous wavelets (in particular, Lemarié-Meyer wavelet). The admissibility property, i.e. $\int_{\mathbb{R}} \psi(t)dt = 0$ is a consequence of the second one and more generally, for all $m \in \mathbb{N}$,

$$\int_{\mathbb{R}} t^m \psi(t)dt = 0. \tag{8}$$

Note that it is not mandatory to choose $\psi$ to be a "mother" wavelet associated to a multiresolution analysis of $\mathbb{L}^2(\mathbb{R})$ and the whole theory can be developed without resorting to this assumption : the choice of $\psi$ is then very large.

Let $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}$, define $\lambda = (a,b)$ and the family of functions $\psi_\lambda$ defined by $\psi_\lambda(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t}{a} - b\right)$. Parameters $a$ and $b$ are so-called the scale and the shift of the wavelet transform (here we consider a continue wavelet transform). Let $d_X(a,b)$ be the wavelet coefficient of the process $X$ for the scale $a$ and the shift $b$, with

$$d_X(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t}{a} - b) X(t) dt = < \psi_\lambda, X >_{L^2(\mathbb{R})} .$$

This family of wavelet coefficients verifies the following properties :

**Property 3.1** *Let $\psi$ verify Assumption (A1) and $X$ be a $(M_K)$-f.B.m. Denote*

$$\mathcal{I}_1(a) = a \int_{\mathbb{R}} |\widehat{\psi}(au)|^2 \, \rho^{-2}(u) \, du. \tag{9}$$

*Then:*

1. *for $a > 0$, $(d_X(a,b))_{b \in \mathbb{R}}$ is a stationary centered Gaussian process such that :*

$$d_X(a,b) = \int_{\mathbb{R}} \frac{\overline{\widehat{\psi_\lambda}(\xi)}}{\rho(\xi)} \overline{\widehat{W}}(d\xi) \quad \text{and} \quad \mathbb{E}\left(d_X^2(a,.)\right) = \mathcal{I}_1(a). \tag{10}$$

2. *for all $i = 0, 1, \cdots, K$, if the scale $a$ is such that $[\frac{\alpha}{a}, \frac{\beta}{a}] \subset [\omega_i, \omega_{i+1}]$, then*

$$\mathbb{E}\left(d_X^2(a,.)\right) = a^{2H_i+1} \sigma_i^2 K_{H_i}(\psi), \quad \text{with } K_H(\psi) = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(u)\right|^2}{|u|^{2H+1}} du. \tag{11}$$

**Remark 3.1** *Another possible interpretation of this property is the following : the process $(d_X(t,0))_{t \in \mathbb{R}_+}$ is a continuous centered Gaussian process $H_i$-self-similar for $t \in [\frac{\beta}{\omega_{i+1}}, \frac{\alpha}{\omega_i}]$, i.e. $(d_X(t,0))_{\beta/\omega_{i+1} \leq t \leq \alpha/\omega_i}$ is a $H_i$-self-similar). However, such a process has no stationary increments.*

The property (11) is useful for the estimation of the parameters of $X$. Indeed, we deduce that if the scale $a \in [\frac{\beta}{\omega_{i+1}}, \frac{\alpha}{\omega_i}]$, then

$$\log\left(\mathbb{E}\left(d_X^2(a,.)\right)\right) = (2H_i + 1) \log a + \log \sigma_i^2 + \log K_{H_i}(\psi). \tag{12}$$

Thus, if we consider a convergent estimator of $\log\left(\mathbb{E}\left(d_X^2(a,.)\right)\right)$, it provides a linear model in $\log a$ and $\log \sigma_i^2$. This natural estimator is $\log I_N(a)$ with

$$I_N(a) = \frac{1}{[N/a]-1}\sum_{k=1}^{[N/a]-1} d_X^2(a, k\Delta_N). \tag{13}$$

This choice is driven by the fact that only $(X(0), X(\Delta_N), \cdots, X(N\Delta_N))$ is known and we suppose that $\psi$ is essentially supported around 0. The stationarity of $d_X^2(a,.)$ implies that for all $a>0$,

$$\mathbb{E}\left(I_N(a)\right) = \mathbb{E}\left(d_X^2(a,.)\right).$$

It is also possible to provide a functional central limit theorem for $(\log I_N(a))_{a_{min}\leq a\leq a_{max}}$ :

**Theorem 3.1** *Let $X$ be a $(M_K)$-f.B.m., $0<a_{min}<a_{max}$ and $\psi$ verify Assumption (A1). Then*

$$\sqrt{N\Delta_N}\left(\log I_N(a) - \log \mathcal{I}_1(a)\right)_{a_{min}\leq a\leq a_{max}} \underset{N\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} (Z(a))_{a_{min}\leq a\leq a_{max}} \tag{14}$$

*with $(Z(a))$ a centered Gaussian process such that for $(a_1,a_2)\in[a_{min},a_{max}]^2$,*

$$cov(Z(a_1),Z(a_2)) = \frac{2a_1\,a_2}{\mathcal{I}_1(a_1)\,\mathcal{I}_1(a_2)}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2 du. \tag{15}$$

Then, if we precise the localization of scales, *i.e.* frequencies, we obtain the interesting following consequence :

**Corollary 3.1** *Let $i\in\{0,1,\cdots,K\}$ and assume that $\dfrac{\beta}{\alpha}\leq\dfrac{\omega_{i+1}}{\omega_i}$. Then,*

$$\sqrt{N\Delta_N}\left(\log I_N(1/f)+(2H_i+1)\log f-\log\sigma_i^2-\log K_{H_i}(\psi)\right)_{\omega_i/\alpha\leq f\leq\omega_{i+1}/\beta} \underset{N\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} (Z(1/f))_{\omega_i/\alpha\leq f\leq\omega_{i+1}/\beta} \tag{16}$$

*with the centered Gaussian process $(Z(.))$ such that for $(f_1,f_2)\in[\dfrac{\omega_i}{\alpha},\dfrac{\omega_{i+1}}{\beta}]^2$,*

$$cov(Z(1/f_1),Z(1/f_2)) = \frac{2\left(f_1\,f_2\right)^{2H_i}}{K_{H_i}^2(\psi)}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{\overline{\widehat{\psi}}(\xi/f_1)\widehat{\psi}(\xi/f_2)}{|\xi|^{2H_i+1}}e^{-iu\xi}d\xi\right)^2 du. \tag{17}$$

For $\Delta_N$ small enough, this result shows that every parameters $H_i$ and $\sigma_i^2$ could be estimate by a linear regression of $\log I_N(1/f_j)$ onto $\log f_j$, when the frequencies $\omega_i$ are known. Moreover, this central limit theorem shows that a graph of $(\log f, \, \log I_N(1/f))$ for $f > 0$ exhibits different areas of asymptotic linearity when $X$ is a $(M_K)$-f.B.m. This purpose will be illustraded in the following example.

As an example of application of the previous corollary, consider a discretized trajectory $(X(\Delta_N), \cdots, X(N.\Delta_N))$ (where $N = 3000$ and $\Delta_N = 0.05$) is obtained from simulation by Cholesky method (that provides L.U. decomposition of the covariance matrix of $(X(\Delta_N), \cdots, X(N.\Delta_N))$ from property (3)).

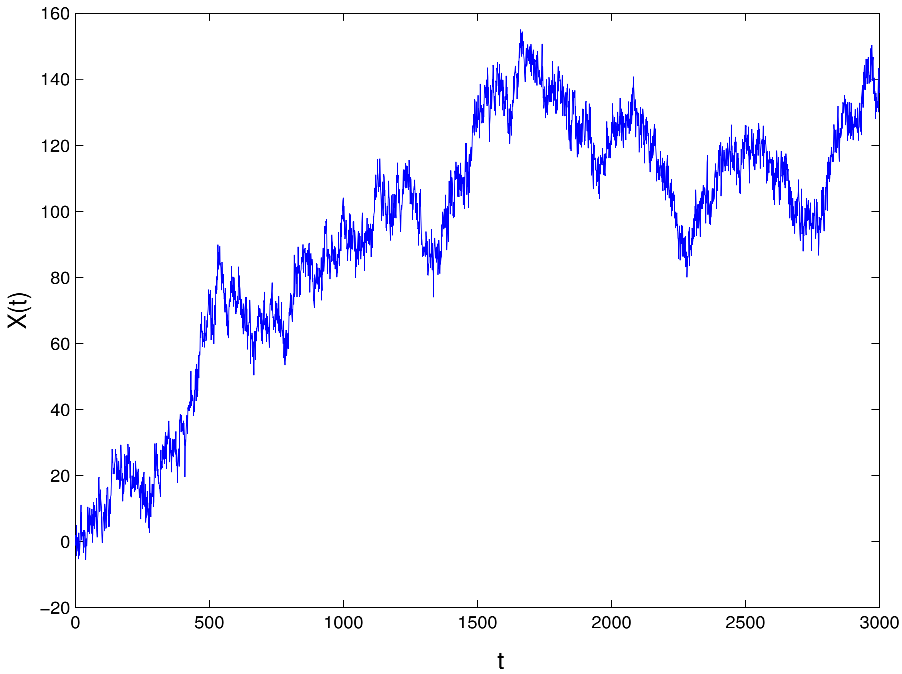


**Figure 1 :** A discretized path $(X(0.05), X(0.10), \cdots, X(150))$ of a $(M_1)$-f.B.m., with $\sigma_0 = \sigma_1 = 5$, $H_0 = 0.6$, $H_1 = 0.2$ and $\omega_1 = 0.5$.

Now it is possible to compute $(\log I_N(f), \log f)$ for numerous values of $f$. More precisely, we define $f_i = f_m q^{i/150}$ for $i = 0, 1, \cdots, 150$ such that $q = \dfrac{f_M}{f_m}$ with $f_m = 0.01$ and $f_M = 10$. The chosen mother wavelet is the Lemarié-Meyer wavelet with $\alpha = 2\pi/3$ and $\beta = 8\pi/3$.

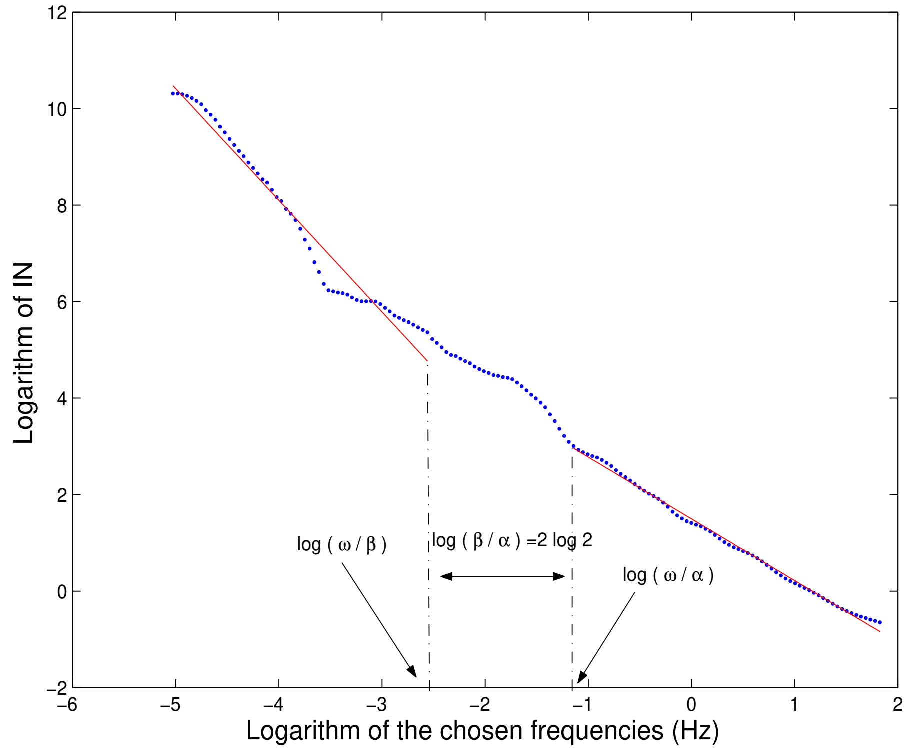


**Figure 2 :** The double logarithm plotting of the variance of wavelet coefficients and its corresponding regression lines for the previous path of a $(M_1)$-f.B.m.

We observe both the different intervals of linearity and obtain the following estimations by linear regression : $\widehat{H}_0 \simeq 0.16$ and $\widehat{H}_1 \simeq 0.65$ that are rather good estimations. This was done with the knowledge of $\omega_1$. In Bardet and Bertrand (2003), we developp a method for estimating $(\omega_i)_i$, the differents parameters $(H_i)_i$ and $(\sigma_i^2)_i$ and construct a goodness-of-fit test for a $(M_K)$-f.B.m.

# A Two examples of applications

## A.1 Finance

Numerous papers in financial economic exhibit the long-range dependence of the price of shares. They are based on statistical studies of financial data, see for instance Willinger *et al.* (1999) and the references therein. These studies have suggested the modeling of log-price process by a fractional Brownian motion. But Rogers (1997) has shown that the f.B.m. is an absurd candidate for the log-price of a share, say $X$, since modeling by f.B.m. ($X = B_H$) would induce the existence of arbitrage opportunities. This mainly follows from the fact that a f.B.m. $B_H$ is not a semi-martingale except when $H = 1/2$. Anyway, the notions should be precise. To be exact, we say that an arbitrage exists if there is some trading strategy whose gain process $(Y_t)_{0\leq t\leq 1}$ satisfies a) $Y_0 = 0 \leq Y_1$, b) $Y_t \geq -1$ for all $t \in [0,1]$, and c) $\mathbb{P}\left(Y_1 > 0\right) > 0$. If the log-price process is not a semi-martingale then there can be no equivalent probability (called risk neutral probability) under which $X$ becomes local martingale. The existence of (at least) one risk neutral probability is equivalent to the NFLVR condition (= *No Free Lunch at Vanishing Risk*), see Delbaen and Schachermayer (1994). The existence of an arbitrage implies the existence of a FLVR, but the converse is false, see counterexamples in Delbaen and Schachermayer (1994). Rogers (1997) shows that when $H \neq 1/2$

1. the f.B.m. $B_H$ is not a semi-martingale, then there exists no risk neutral probability,

2. there exists an arbitrage.

This study is based on the moving average representation

$$X_\varphi(t) = C_\varphi^{-1} \int_{-\infty}^{+\infty} \left[\varphi(t-s) - \varphi(-s)\right] dW_s, \tag{18}$$

with the kernel $\varphi(x) = |x|^{H-1/2}\, \mathbf{1}_{[0,+\infty[}(x)$. But Rogers (1997) remarked that the existence of arbitrage is linked to the behaviour of the kernel at the vicinity of 0.

He proposed to replace the kernel by a regularized kernel (in the vicinity of 0), then the process $X$ defined by (18) becomes a semi martingale and exhibit the same long-range dependence as f.B.m. Cheridito (2000) gives a class of regularized kernel for which the corresponding price process is arbitrage-free and also equivalent to Brownian motion, therefore there exists a unique risk neutral probability and he derives formulas for the price of European option.

The regularization of the kernel $\varphi$ near 0 corresponds to a modification of the high frequencies behaviour of the process. The case where $X$ is a $(M_K)$-f.B.m. corresponds to another modification. Formula (4) shows that a $(M_K)$-f.B.m. is a semi martingale if and only if $H_K = 1/2$. Let us precise the reasoning, $X$ is a semi martingale iff $B_{H_K}$ is a semi martingale and the f.B.m. $B_{H_K}$ is a semi martingale if and only if $H_K = 1/2$, see Rogers (1997). In conclusion, the $(M_K)$-f.B.m. with $H_K = 1/2$ appears as a reasonable model for the log-price process, this is confirm by the statistical study given in Bardet (2000). The pricing of European option with this model remains open.

## A.2 Biomechanical applications

In Biomechanic, one wishes to model the upright position of human being in quiet stance and understand the control system maintaining the upright posture. The position of the center of pressure (C.O.P.) is measured on a force platform during one minute at 100 Hz, leading to set of 6000 data. Collins and de Luca (1993) [10] introduced the use of f.B.m. for modeling these data. From the application of the log-variogram method, Collins and de Luca interpret the obtained graph as corresponding to a f.B.m. with two regimes *a short term* with slope $2H_1$ and *a long term* with slope $2H_0$ separated by a critical time lag $\delta_c$ and these parameters are estimated graphically : one obtains $H_0 > 0.5$, $H_1 < 0.5$ and a critical time lag $\delta_c \simeq 1\ s$. In fact, the first analysis would suggest the modeling by a $(M_1)$-f.B.m. This method has been used many times in biomechanics with different experimental conditions (opened eyes versus closed eyes, different feet angles,...). But the automatic determination of the critical time lag $\delta_c$ (corresponding to the frequency change $\omega_1$) and of the two slopes $2H_0$, $2H_1$ is problematic with the statistic based on the variogram.

Here, we present the result of the wavelet analysis for a lateral path of the C.O.P. The chosen mother wavelet is such that $\alpha = 5$ and $\beta = 10$ and the frequency change is automatically detected. We refer to Bardet and Bertrand (2003) for a complete study of biomechanical data and its conclusions.

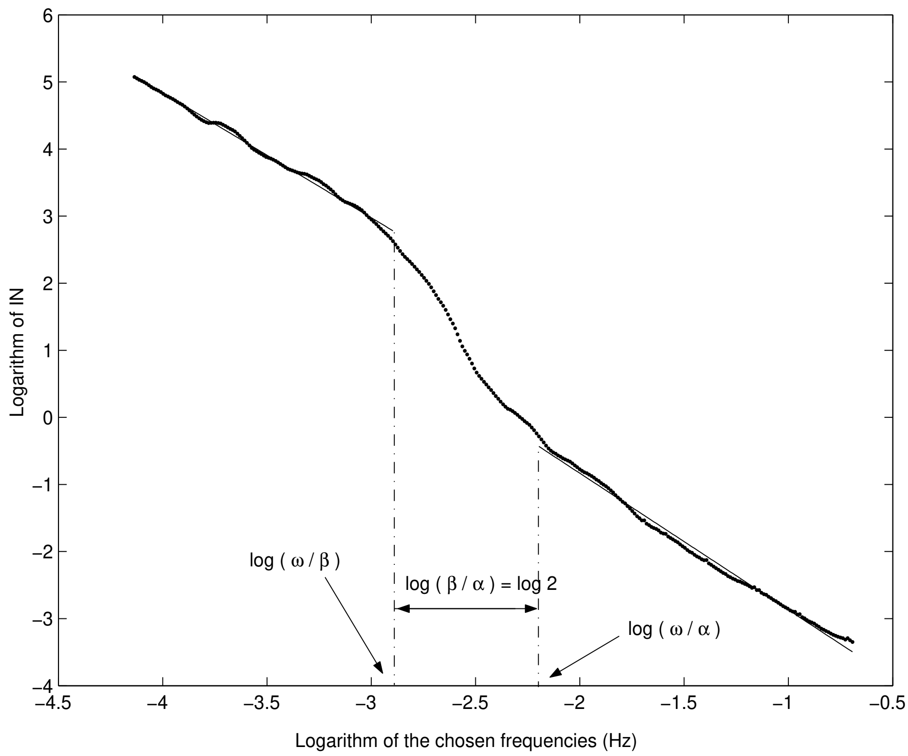


**Figure 2 :** The double logarithm plotting of the variance of wavelet coefficients and its corresponding regression lines for a lateral trajectory of the C.O.P.[1]

[1] these experimental data were realized by A. Mouzat and are used in [7].

# B Proofs

## B.1 Proofs of section 2

**Proof.** [**Property 2.1**] **1.** See Cramér and Leadbetter (1967) for the proof of the stationarity of the increments of $X$.

**2.** We easily obtain :

$$\begin{aligned}\mathcal{V}(\delta) &= \int_{I\!R} \frac{|e^{i\delta\xi}-1|^2}{\rho^2(\xi)} d\xi \;=\; 4\int_0^{+\infty} \frac{(1-\cos(\delta\xi))}{\rho^2(\xi)} d\xi \\ &= 4\sum_{i=0}^{K} \sigma_i^2 \int_{\omega_i}^{\omega_{i+1}} \frac{(1-\cos(\delta\xi))}{\xi^{2H_i+1}} d\xi,\end{aligned}$$

that provides us (3) after the change of variable $\delta\xi = v$.

**3.** Formula (2) gives us

$$\begin{aligned}X(t) &= \int_{I\!R} \frac{\left(e^{it\xi}-1\right)}{|\xi|^{H_K+1/2}} \widehat{W}(d\xi) \\ &- \int_{|\xi|\le\omega_K} \frac{\left(e^{it\xi}-1\right)}{|\xi|^{H_K+1/2}} \widehat{W}(d\xi) \;+\; \sum_{j=0}^{K-1} \int_{\omega_j<|\xi|\le\omega_{j+1}} \frac{\left(e^{it\xi}-1\right)}{|\xi|^{H_j+1/2}} \widehat{W}(d\xi)\end{aligned}$$

The harmonizable representation (1) shows that $\int_{I\!R} \frac{\left(e^{it\xi}-1\right)}{|\xi|^{H_K+1/2}} \widehat{W}(d\xi)$ is a f.B.m. with Hurst index $H_K$, we call it $B_{H_K}$, this provides the decomposition (4) with

$$R(t) \;=\; Z_{H_K,\omega_K}(t) \;+\; \sum_{j=0}^{K-1} \left(Z_{H_j,\omega_{j+1}}(t) - Z_{H_j,\omega_j}(t)\right)$$

where by definition

$$Z_{H,\omega}(t) \;=\; \int_{|\xi|\le\omega} \frac{\left(e^{it\xi}-1\right)}{|\xi|^{H+1/2}} \widehat{W}(d\xi).$$

When $H \in ]0,1[$ and $\omega \in ]0,+\infty[$, the process $Z_{H,\omega}$ has finite variation. Since $R(t)$ is a finite sum of these processes, it is a process with finite variation. Therefore, the decomposition (4) induces that the regularity of $X$ is exactly the regularity of the f.B.m. with parameter $B_{H_K}$. This finishes the proof of Property 2.1. ■

**Proof.** [**Property 2.2**] First, we have

$$r(n) = 8\sum_{i=0}^{K} \sigma_i^2 \int_{\omega_i}^{\omega_{i+1}} \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_i+1}}\, d\xi. \tag{19}$$

Indeed, using (2) and the definition of $r(n)$, we get :

$$\begin{aligned} r(n) = \mathrm{cov}\left(Y_\rho(n), Y_\rho(0)\right) &= \mathrm{cov}\left[\left(X(n+1)-X(n)\right), \left(X(1)-X(0)\right)\right] \\ &= \int_{I\!R} \frac{\left(e^{i\xi(n+1)}-e^{i\xi n}\right)}{\rho(\xi)} \frac{\left(e^{-i\xi}-1\right)}{\rho(\xi)}\, d\xi \\ &= \int_{I\!R} e^{in\xi} \frac{\left|e^{-i\xi}-1\right|^2}{\rho^2(\xi)}\, d\xi \\ &= \int_0^{+\infty} \frac{\cos(n\xi)\sin^2(\xi/2)}{\rho^2(\xi)}\, d\xi, \end{aligned}$$

which induces (19) by definition of the $(M_k)$-f.B.m. Formulae (5) and (6) are asymptotic expansion of $r(n)$ following the powers of $1/n$. After two integrations by parts, we have the following calculations : for every $\omega > 0$ and $H \in ]0,1[$

$$\begin{aligned} &\int_\omega^\infty \frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H+1}} d\xi = \\ &= \frac{1}{n}\left[\frac{-\sin(n\omega)\sin^2(\omega/2)}{\omega^{2H+1}} - \int_\omega^\infty \sin(n\xi)\left(\frac{\sin\xi}{2\xi^{2H+1}} - \frac{(2H+1)\sin^2(\xi/2)}{\xi^{2H+2}}\right) d\xi\right] \\ &= -\frac{1}{n}\left(\frac{\sin(n\omega)\sin^2(\omega/2)}{\omega^{2H+1}}\right) - \frac{1}{n^2}\cos(n\omega)\left(\frac{\sin\omega}{\omega^{2H+1}} - (2H+1)\frac{\sin^2(\omega/2)}{\omega^{2H+2}}\right) \\ &\quad + \frac{1}{n^2}\int_\omega^\infty \cos n\xi\left(\frac{\cos\xi}{\xi^{2H+1}} - (2H+1)\frac{\sin(\xi)}{\xi^{2H+2}} + (2H+1)(2H+2)\frac{\sin^2(\xi/2)}{\xi^{2H+2}}\right) d\xi \end{aligned}$$

For $\omega > 0$ and $H > 0$, the last integral is bounded by

$$F \times \int_\omega^\infty \left[\xi^{-(2H+1)} + \xi^{-(2H+2)}\right] \le F \times \left[\omega^{-2H} + \omega^{-(2H+1)}\right] < +\infty,$$

with $F>0$. This induces for every $\omega>0$ and $H\in]0,1[$

$$\int_{\omega}^{\infty}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H+1}}d\xi \quad = \quad -\frac{1}{n}\left(\frac{\sin(n\omega)\sin^2(\frac{\omega}{2})}{\omega^{2H+1}}\right)+\mathcal{O}\left(\frac{1}{n^2}\right). \tag{20}$$

We immediately deduce from (20) that for $i=1,2,..,K$, with $\omega_{K+1}=\infty$, we have

$$\int_{\omega_i}^{\omega_{i+1}}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_i+1}}d\xi = \tag{21}$$

$$= \frac{1}{n}\left(\frac{\sin(n\omega_{i+1})\sin^2(\frac{\omega_{i+1}}{2})}{\omega_{i+1}^{2H_i+1}}-\frac{\sin(n\omega_i)\sin^2(\frac{\omega_i}{2})}{\omega_i^{2H_i+1}}\right)+\mathcal{O}\left(\frac{1}{n^2}\right).$$

¿From the other hand, consider the first term in the sum in the right hand side of (19). We have

$$\int_0^{\omega_1}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}}d\xi =$$

$$= \int_0^{+\infty}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}}d\xi-\int_{\omega_1}^{+\infty}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}}d\xi. \tag{22}$$

The first integral correspond to the case of a constant Hurst parameter, that is the usual f.B.m. Let $X_0$ be a f.B.m. with parameter $H_0$ and variance 1, defined by

$$X_0(t)=\frac{1}{C(H_0)}\int_{I\!R}\frac{(e^{it\xi}-1\ )}{|\xi|^{H_0+1/2}}\overline{\widehat{W}}(d\xi)\quad\text{for } t\in I\!R_+$$

with $C(H_0)=\pi^{1/2}(H_0\Gamma(2H_0)\sin(\pi H_0))^{-1/2}$, then we have

$$\mathrm{cov}\left((X_0(n+1)-X_0(n),X_0(1)-X_0(0)\right)=\frac{1}{C^2(H_0)}\int_0^{\infty}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}}d\xi$$

and (see [21] p. 335) we have the following expansion

$$\mathrm{cov}\left((X_0(n+1)-X_0(n),X_0(1)-X_0(0)\right)=H_0(2H_0-1)\frac{1}{n^{2-2H_0}}+\mathcal{O}(\frac{1}{n^{3-2H_0}}).$$

Combined with (20) and (22), for all $H_0\in]0,1[$, we have

$$\int_0^{\omega_1}\frac{\cos(n\xi)\sin^2(\xi/2)}{\xi^{2H_0+1}}d\xi \quad =$$

$$= \left(\frac{\pi(2H_0-1)}{r(2H_0)\sin(\pi H_0)}\right)\frac{1}{n^{2-2H_0}}+\left(\frac{\sin(n\omega_1)\sin^2(\frac{\omega_1}{2})}{\omega_1^{2H_0+1}}\right)\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}+\frac{1}{n^{3-2H_0}}\right) \tag{23}$$

As a consequence from (21) and (23), for $H_0 > 1/2$, we obtain (5) and if $0 < H_0 \leq 1/2$,

$$
\begin{aligned}
&\operatorname{cov}\left(Y_\rho(n)Y_\rho(0)\right) \\
&= \frac{8}{n}\left[\sum_{i=1}^{K-1}\sigma_i^2\left(\frac{\sin(n\omega_{i+1})\sin^2(\frac{\omega_{i+1}}{2})}{\omega_{i+1}^{1+2H_{i+1}}} - \frac{\sin(n\omega_i)\sin^2(\frac{\omega_i}{2})}{\omega_i^{1+2H_i}}\right)\right. \\
&\quad \left. + \sigma_0^2\left(\frac{\sin(n\omega_1)\sin^2(\frac{\omega_1}{2})}{\omega_1^{2H_0+1}}\right) \; - \; \sigma_K^2\frac{\sin(n\omega_K)\sin^2(\frac{\omega_K}{2})}{\omega_K^{2H_K+1}}\right] \\
&\quad + \frac{\pi\sigma_0^2(2H_0-1)}{\Gamma(2H_0)\sin(\pi H_0)}\frac{1}{n^{2-2H_0}} \; + \; \mathcal{O}(\frac{1}{n^2}),
\end{aligned}
$$

and (6) is proved. ■

## B.2 Proofs of section 3

**Proof.** [**Property 3.1**] First we have

$$
\begin{aligned}
d_X(\lambda) &= \int_{\mathbb{R}}\psi_\lambda(t)\left[\int_{\mathbb{R}}\frac{\left(e^{it\xi}-1\right)}{\rho(\xi)}\overline{\widehat{W}}(d\xi)\right]dt \\
&= \int_{\mathbb{R}}\frac{1}{\rho(\xi)}\left[\int_{\mathbb{R}}\left(e^{it\xi}-1\right)\psi_\lambda(t)dt\right]\overline{\widehat{W}}(d\xi)
\end{aligned}
$$

But $\psi$ verifies $\int\psi = 0$ and it implies (10). Now, we prove that $(d_X(a,b))_{b\in\mathbb{R}}$ is a stationary centered Gaussian process. The only problem is the stationarity. It is obvious that

$$
\overline{\widehat{\psi_\lambda}}(\xi) = \sqrt{a}\times e^{-iab\xi}\times\overline{\widehat{\psi}}(a\xi), \tag{24}
$$

Combined with (10), we deduce that for $a > 0$ and $(b,b') \in \mathbb{R}^2$, we have:

$$
\mathbb{E}\left(d_X(a,b)d_X(a,b')\right) \;=\; \sqrt{a\,a'}\int_{\mathbb{R}}e^{-ia(b-b')\xi}\left|\frac{\widehat{\psi}(a\xi)}{\rho(\xi)}\right|^2 d\xi. \tag{25}
$$

Thus, for a given $a > 0$, $\mathbb{E}\left(d_X(a,b)d_X(a,b')\right)$ is only depending on $(b-b')$ which induces that $(d_X(a,b))_{b\in\mathbb{R}}$ is a stationary process.

The second part of the property is a consequence of this first part. In fact, when $b=b'$, the scale $a$ verifies $[\frac{\alpha}{a},\frac{\beta}{a}]\subset]\omega_i,\omega_{i+1}[$ and $\psi$ verifies Assumption (A1), formula (25) implies

$$\begin{aligned}
\mathbb{E}\left(d_X^2(a,.)\right) &= 2a\int_{\alpha/a}^{\beta/a}\left|\frac{\overline{\widehat{\psi}}(av)}{\rho(v)}\right|^2 dv\\
&= 2a\int_{\alpha/a}^{\beta/a}\left|\widehat{\psi}(a\,v)\right|^2\sigma_i^2\,v^{-(2H_i+1)}dv\\
&= a\sigma_i^2\int_{\mathbb{R}}\left|\widehat{\psi}(av)\right|^2 v^{-(2H_i+1)}dv\\
&= a^{2H_i+1}\sigma_i^2\,K_{H_i}(\psi),
\end{aligned}$$

and the second part of the property is proved. ■

**Proof.** [**Proposition 3.1**] (Both first steps follow the same procedure that in Bardet, 2002).

**First step :** For $(a,a')\in]0,\infty[^2$, and $(b,b')\in\mathbb{R}^2$, using (24) we get :

$$\begin{aligned}
\left|\mathbb{E}\left(d_X(a,b)d_X(a',b')\right)\right| &= \sqrt{aa'}\int_{\mathbb{R}} e^{-i(ab-a'b')\xi}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}d\xi\\
&\leq \frac{C}{(ab-a'b')^2},
\end{aligned}\tag{26}$$

when $|ab-a'b'|>1$. The last inequality follows from assumption (A1) and a double integration by part (with $C>0$). By this way, and with:

$$\begin{aligned}
\operatorname{cov}\left(I_N(a)I_N(a')\right) &= \frac{1}{[N/a][N/a']}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\operatorname{cov}\left(d_X^2(a,p\Delta_N)d_X^2(a',p'\Delta_N)\right)\\
&= \frac{2}{[N/a][N/a']}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\operatorname{cov}^2\left(d_X(a,p\Delta_N)d_X(a',p'\Delta_N)\right),
\end{aligned}$$

because $d_X(.,.)$ is a Gaussian random variable. Define $\lambda_N(a,a')$ the set of pair $(p,p')$ such that $|ap\Delta_N - a'p'\Delta_N| < 1$, then for $(p,p') \in \lambda_N(a,a')$ we have

$$\mathrm{cov}^2\left(d_X(a,p\Delta_N)d_X(a',p'\Delta_N)\right) \leq \mathbb{E} d_X^2(a,.)\mathbb{E} d_X^2(a',.) \leq M(a,a')$$

with $M(a,a') > 0$. From the other hand, for a pair $(p,p')$ satisfying $|ap\Delta_N - a'p'\Delta_N| \geq 1$, formula (26) induces that

$$\mathrm{cov}^2\left(d_X(a,p\Delta_N)d_X(a',p'\Delta_N)\right) \leq C^2|ap\Delta_N - a'p'\Delta_N|^{-4}.$$

**Second step :** we get

$$\begin{aligned}
\mathrm{cov}\left(I_N(a)I_N(a')\right) &= \frac{2}{[N/a][N/a']}\left(\sum_{(p,p')\in\lambda_N(a,a')} \mathrm{cov}^2\left(d_X^2(a,p\Delta_N)d_X^2(a',p'\Delta_N)\right) + ..\right.\\
&\qquad \left. .. + \sum_{(p,p')\notin\lambda_N(a,a')} \mathrm{cov}^2\left(d_X^2(a,p\Delta_N)d_X^2(a',p'\Delta_N)\right)\right)\\
&\leq \frac{2}{[N/a][N/a']}\left(M\,|\lambda_N(a,a')|+\frac{C'}{\Delta_N^2}\iint_{|at-a't'|\geq 1}\frac{\mathbf{1}_{0\leq\Delta_N t\leq N/a}\mathbf{1}_{0\leq\Delta_N t'\leq N/a'}}{|at-a't'|^4}dtdt'\right)\\
&\leq \frac{2}{[N/a][N/a']}\left(\frac{2MN}{aa'\Delta_N}+\frac{C'}{\Delta_N^2 a}\int_0^{\Delta_N N/a'}\int_1^N\frac{du}{u^4}dt'\right)\\
&\leq \frac{M'}{N\Delta_N}+\frac{C''}{N\Delta_N}.
\end{aligned}$$

This induces that $N\Delta_N \mathrm{cov}\left(I_N(a)I_N(a')\right) \to \gamma_\rho(a,a')$ when $N\to\infty$, with $\gamma_\rho(a,a') > 0$. More precisely, let $f(t,t') = \displaystyle\int_{\mathbb{R}} \frac{\overline{\widehat{\psi}(a\xi)}\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(at-a't')\Delta_N\xi}d\xi$. Then, $f^2 \in \mathcal{C}^\infty(\mathbb{R}^2)$ because $\psi$ verifies Assumption (A1). More, for $(k,k') \in \mathbb{N}^*\times\mathbb{N}^*$,

$$\left|\int_{k-1}^{k}\int_{k'-1}^{k'}\left(f^2(t,t')-f^2(k,k')\right)dt\right| \leq 2\sup_{(t,t')\in\mathbb{R}^2}\left|\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial t'}\right)(t,t')\right||f(\theta_k,\theta_{k'}|,$$

with $k-1 \leq \theta_k \leq k$ and $k'-1 \leq \theta_{k'} \leq k'$. From obvious computations, it appears that $2\displaystyle\sup_{(t,t')\in\mathbb{R}^2}\left|\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial t'}\right)(t,t')\right| \leq C\,\Delta_N$ with $C$ a positive constant depending only on $\psi$, $\rho$, $a_{min}$ and $a_{max}$. Moreover, from the previous calculations

$\displaystyle\frac{2aa'\Delta_N}{N}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}|f(\theta_p,\theta_{p'}|<C'$. As a consequence,

$$N\Delta_N \mathrm{cov}\,(I_N(a)I_N(a'))=\frac{2aa'\Delta_N}{N}\sum_{p=1}^{[N/a]}\sum_{p'=1}^{[N/a']}\left(\sqrt{aa'}\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(ap-a'p')\Delta_N\xi}d\xi\right)^2$$
$$=\frac{2a^2a'^2}{N\Delta_N}\int_{t=0}^{[N/a]\Delta_N}\int_{t'=0}^{[N/a']\Delta_N}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(at-a't')\xi}d\xi\right)^2 dtdt'+\mathcal{O}(\Delta_N)\quad(27)$$

Thus,

$$\begin{aligned}\gamma_\rho(a,a') &= \lim_{N\to\infty}N\Delta_N\mathrm{cov}\,(I_N(a)I_N(a'))\\
&= \lim_{N\to\infty}\frac{2a^2a'^2}{N\Delta_N}\int_{t=0}^{[N/a]\Delta_N}\int_{t'=0}^{[N/a']\Delta_N}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(at-a't')\xi}d\xi\right)^2 dtdt'\\
&= \lim_{N\to\infty}\frac{2aa'}{N\Delta_N}\int_{0}^{a[N/a]\Delta_N}\int_{0}^{a'[N/a']\Delta_N}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-i(t-t')\xi}d\xi\right)^2 dtdt'\\
&= \lim_{N\to\infty}\frac{2aa'}{N\Delta_N}\int_{-N\Delta_N}^{N\Delta_N}(N\Delta_N-|u|)\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2 du\\
&= \lim_{N\to\infty}\frac{2aa'}{N\Delta_N}\left[\int_{|u|\le N\Delta_N}(N\Delta_N)\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2+\cdots\right.\\
&\qquad\left.\cdots+\int_{|u|\le N\Delta_N}(-|u|)\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2 du\right]\\
&= 2aa'\int_{I\!R}\left(\int_{I\!R}\frac{\overline{\widehat{\psi}}(a\xi)\widehat{\psi}(a'\xi)}{|\rho(\xi)|^2}e^{-iu\xi}d\xi\right)^2 du+\lim_{N\to\infty}\frac{2aa'}{N\Delta_N}\mathcal{O}(1),\end{aligned}$$

from the previous bound of $\gamma_\rho(a,a')>0$. As a consequence, we obtain the following central limit theorem for all $m\in I\!N^*$ and $(a_j)_{1\le m}\in[a_{min},a_{max}]^m$ :

$$\sqrt{N\Delta_N}\left(I_N(a_j)-a_j\int_{I\!R}\left|\frac{\overline{\widehat{\psi}}(a_j\xi)}{\rho(\xi)}\right|^2 d\xi\right)_{1\le j\le m}\xrightarrow[N\to\infty]{\mathcal{D}}\mathcal{N}_m(0,\Gamma),$$

with $\Gamma = (\gamma_{k,l})_{1\leq k,l\leq m}$ and

$$\gamma_{k,l} = 2a_k a_l \int_{I\!R} \left( \int_{I\!R} \frac{\overline{\widehat{\psi}}(a_k\xi)\widehat{\psi}(a_l\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi \right)^2 du.$$

**Third step :** Now, we have to prove the tightness of $\sqrt{N\Delta_N}\,(I_N(a) - I\!E I_N(a))_{a_{min}\leq a\leq a_{max}}$. Let $a_{max} \leq a_1 < a < a_2 \leq a_{min}$ and $L_N(a) = \sqrt{N\Delta_N}\,(I_N(a) - I\!E I_N(a))$. Then,

$$\begin{aligned} I\!E\,|L_N(a) - L_N(a_1)|^2 &= N\Delta_N\,(\mathrm{var} I_N(a) + \mathrm{var} I_N(a_1) - 2\mathrm{cov}(I_n(a), I_N(a_1))) \\ &= \gamma_\rho(a,a) + \gamma_\rho(a_1,a_1) - 2\gamma_\rho(a,a_1) + \frac{C''(N,a,a_1)}{N\Delta_N}, \end{aligned}$$

from (27) and with $|C''(N,a,a_1)| \leq C(\psi,\rho,a_{min},a_{max})$ (see previously). But $(a,a_1) \mapsto \gamma_\rho(a,a_1)$ is $\mathcal{C}^\infty([a_{min},a_{max}]^2)$ because $\psi$ verifies Assumption (A1). Then, for $N$ large enough it exists a positive constant $D$, independent of $a$ and $a_1$ such that :

$$I\!E\,|L_N(a) - L_N(a_1)|^2 \leq D|a - a_1|.$$

As a consequence, from Cauchy-Schwartz inequality,

$$I\!E\,|L_N(a) - L_N(a_1)|\,|L_N(a_2) - L_N(a)| \leq D\sqrt{|a - a_1||a_2 - a|},$$

and thus the random function $L_N(a)$, $N = 1, 2, \cdots,$, of $\mathcal{D}([a_{min}, a_{max}])$ are tight by Theorem 15.4 and 15.6 of Billingsley (1968).

**Fourth step :** The function $x \mapsto \log x$ is $\mathcal{C}^\infty$ on $I\!R_+^*$. Thus, from the Slutsky theorem, Proposition 3.1 is proved. ∎

# References

[1] Abry, P., Flandrin, P., Taqqu, M. and Veitch, D. , 2002. Self-similarity and long-range dependence through the wavelet lens, in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser, p. 527-556..

[2] Bardet J.M., 2000. Les cours d'actifs financiers sont-ils autosimilaires? *Journal de la SFdS* 141, p. 137-148.

[3] Bardet J.M., 2002. Statistical study of the wavelet analysis of fractional Brownian motion. *IEEE Trans. Inform. Theory.* 48, p. 991-999.

[4] Bardet J.M. and Bertrand, P., 2003. Identification of the multiscale fractional Brownian motion with biomechanical applications. Preprint of L.S.P., Université Toulouse III.

[5] Bardet J.M., Lang G., Moulines E. and Soulier P., 2000. Wavelet estimator of long-range dependent processes. *Statistical Inference for Stochastic Processes* 3, p. 85-99.

[6] Benassi, A. and Deguy, S., 1999. Multi-scale fractional Brownian motion : definition and identification. Preprint LAIC.

[7] Bertrand, P., Bardet, J.M, Dabonneville, M., Mouzat, A. and Vaslin, P., 2001. Automatic determination of the different control mechanisms in upright position by a wavelet method, *I.E.E.E. Engineering in Medicine and Biology Society.*

[8] Billingsley, P., 1968. *Convergence of Probability Measures.* New-York. Wiley.

[9] Cheridito, P., 2000. Regularized Fractional Brownian Motion and Option Pricing. Preprint.

[10] Collins J.J. and De Luca C.J., 1993. Open-loop and closed-loop control of posture : A random walk analysis of center-of-pressure trajectories. *Experimental Brain Research* 9, p. 308-318.

[11] Cramér, H.and Leadbetter, M. R., 1967. *Stationary and related stochastic processes. Sample function properties and their applications.* Wiley and Sons.

[12] Dahlhaus R., 1989. Efficient parameter estimation for self-similar processes. *Ann. Statist.* 17, p. 1749-1766

[13] Delbaen, F. and Schachermayer, W., 1994. "A general version of the fundamental theorem of asset pricing", *Math. Annals*, (1994) p. 463-520.

[14] Flandrin, P., 1992. Wavelet analysis and synthesis of fractional Brownian motion. *IEEE Trans. on Inform. Theory* 38, p. 910-917.

[15] Fox, R. and Taqqu, M.S., 1986. Large-sample properties of parameter estimates for strongly dependent Gaussian time series. *Ann. Statist.* 14, p. 517-532.

[16] Frisch, U., 1995. *Turbulence, the legacy of A.N. Kolmogorov*, Cambridge University Press.

[17] Giraitis L. and Surgailis D., 1990. A central limit theorem for quadratic forms in strongly dependent linear variables and its applications to the asymptotic normality of Whittle estimate. *Prob. Th. and Rel. Fields.* 86, p. 87-104.

[18] Mandelbrot, B. and Van Ness J., 1968. Fractional Brownian motion, fractional noises and applications. *SIAM review* 10, p.422-437.

[19] Papanicolaou, G. and Solna, K., 2002. Wavelet Based Estimation of Local Kolmogorov Turbulence, , in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser, p. 473-506

[20] Rogers, L.C.G., 1997. Arbitrage with fractional Brownian motion. *Mathematical Finance* 7, p. 95-105.

[21] Samorodnitsky, G. and Taqqu M. S., 1994. *Stable non-Gaussian Random Processes*, Chapman & Hall.

[22] Willinger, W., Taqqu,M.S. and Teverovsky, V., 1999. Stock market price and long-range dependence, *Finance and Stochastics*, 1, p. 1-14