Bivariate occupation measure dimension of multidimensional processes

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Abstract. Bivariate occupation measure is a new dimension for multidimensional continuous random processes. This dimension is given by the asymptotic behavior of its bivariate occupation measure. We compare first this dimension to the Hausdorff dimension. Secondly, we study relations between these dimensions and the existence of local time or self-intersection local time of the process. Finally, we compute the local correlation dimension of multidimensional Gaussian and stable processes with local Hölder properties and show it is the same value that the Hausdorff dimension of its image. By the way, we give a new a.s. convergence of the bivariate occupation measure of a multidimensional fractional Brownian or particular stable motion (and thus of a spatial Brownian or Levy stable motion).

1 Introduction

We consider a $\mathbb{R}^d$-valued continuous random multidimensional process $X = \{X_t; t \in \mathbb{R}\}$ and for J a compact set of $\mathbb{R}^2$ we define its bivariate occupation measure $\mu(I,J)$ in a compact set $I \in \mathbb{R}^d$, i.e.

$$\mu(I,J) = \int_J \mathbb{I}_{(X_t - X_s \in I)} ds dt.$$

If $\mu(.,J)$ is absolutely continuous with respect to $\mathbb{R}^d$-Lebesgue measure λ_d then X has a self-intersection local time (S.I.L.T.) denoted $\alpha(.,J)$ which is defined by the Radon-Nikodym derivative $\dfrac{d\mu}{d\lambda_d}$. This definition of S.I.L.T. was first established by Rosen in [17] and [18]. An interesting consequence is the study of the Hausdorff dimension of multiple points of Gaussian fields (another results and methods are given by Cuzick [5]). But if J contains the diagonal $\{s = t\}$ then lots of random processes do not have a S.I.L.T. However, Varadhan's renormalization technic can be used to study the behavior of $\alpha(x,J)$ when $x \to 0$ (see Rosen [20]) or of $\alpha_\varepsilon(0,J)$ when $\varepsilon \to 0$, where $\alpha_\varepsilon(.) = \displaystyle\int_J \delta_\varepsilon(X_t - X_s) ds dt$ and δ_ε is a sequence of regularizing kernels (see Le Gall [13] or Rosen [19]).

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In this paper, we have chosen to work with $\mu(I,J)$ to avoid these difficulties. In fact, we study the asymptotic behavior of $\mu(B_d(0,r),[0,T]^2)$ when $r\to 0$, with $T>0$ and $B_d(0,r)$ a d-ball with center 0 and radius r. It is well known that if it exists $C>0$ such that $\mu(B_d(0,r),[0,T]^2)\sim Cr^d$ when $r\to 0$ then X has a S.I.L.T. in 0 (see Geman and Horowitz [9], Theorem 7.2). Now, let's assume for a path of X that :

$$\mu(B_d(0,r),[0,T]^2)\sim r^\nu L(r)\ \text{ when } r\to 0,$$

where $\nu>0$ and $L(.)$ is a slow varying function such that verifies $\lim\limits_{r\to 0}\log L(r)/\log r=0$. First, we prove that ν is smaller than or equal to Hausdorff dimension of this path. Secondly, we show that if X has a local time (L.T.) for all $x\in \mathbb{R}^d$ that verifies weak conditions or if X has a positive continuous S.I.L.T. in 0 then ν exists and $\nu=d$. As a consequence, we give new relations between the existence of a L.T. or a S.I.L.T. for X (or a sub-process of X) and the value of the Hausdorff dimension of the image of X. Finally, ν has all the properties generally required to define a dimension and we call it (whenever it exists) bivariate occupation measure dimension.

The main results are given in Section 3 which is devoted to continuous multidimensional stable processes with index $(\alpha_1,\alpha_2,...,\alpha_d)$ where each $\alpha_i\in(0,\max(1,p_i^{-1})]$ and by convention $\alpha_1\le\cdots\le\alpha_d$ (for an introduction to these processes, see Adler, 1981, in the Gaussian case and Nolan, 1988, or Samorodnitsky and Taqqu, 1994, in the stable case). Each α_i corresponds to the local Hölder exponent of the i-th coordinate of X. In such a case, Theorem 3.1 yields that the bivariate occupation measure dimension ν exists and

$$\nu=\min\left(d,\min_{i\in\{1,..,d\}}\left\{\frac{1+\sum_{k=1}^{i}(\alpha_i-\alpha_k)}{\alpha_i}\right\}\right)\quad a.S.$$

This value of ν is also the Hausdorff dimension of the image of X (see results of Cuzick [4] corrected by Xiao [22]).
We also establish new a.s. convergences of bivariate occupation measures for particular cases of these processes (for instance, spatial Brownian or fractional Brownian motions, and L/'evy or linear fractional stable motions). In fact, we show that if $\alpha_1+\alpha_2+...+\alpha_d>1$ then with $C>0$,

$$\mu(B_d(0,r),[0,T]^2)=\int_0^T\int_0^T \mathbb{I}_{(\|X_t-X_s\|\le r)}\sim Cr^\nu\left(\log\frac{1}{r}\right)^e\quad \text{when } r\to 0 \text{ a.s.},$$

with $e=1$ when $\nu\in\{1,2,..,d-1\}$ (and if $\alpha_\nu\neq\alpha_{\nu+1}$) or else $e=0$. If $\alpha_1+\alpha_2+...+\alpha_d=1$, we also show that it exists $C'>0$ such that for r small enough, $\mu(B_d(0,r),[0,T]^2)\ge C'r^d\log\dfrac{1}{r}$ a.s..

There are two advantages of the introduction of this new dimension. First, it provides new connections between the existence of L.T. and the value of the Hausdorff dimension of the image of a process. As a consequence, a new explanation of this value is obtained. Secondly, it provides a new estimation of the Hausdorff dimension. In fact, if a discretized trajectory $(X_{\frac{T}{N}}(\omega),X_{2\frac{T}{N}}(\omega),...,X_T(\omega))$ of a multidimensional continuous process is known, with distinct

$(r_i)_{1\leq i\leq k}$ positive and small enough real numbers, an estimation of the Hausdorff dimension of the image of X is provided from a linear regression of $\log C(N, r_i)$ by $\log r_i$ where

$$C(N,r) = \frac{2}{N(N-1)} \sum_{1\leq i<j\leq N} \mathrm{I\!I}_{(\|X_{iT/N}-X_{jT/N}\|\leq r)}.$$

By the way, the bivariate occupation measure dimension appears as a local correlation dimension (see Bardet [2]). Usual correlation dimension was introduced by Grassberger and Proccacia [10] for easily computing Hausdorff dimension of strange attractors (see Eckmann and Ruelle [6]).

2 Definitions and properties

In all this section, we consider a $\mathbb{R}^d$-valued continuous random process $X = \{X_t, t \in [0,T]\}$ defined on the measurable space $(\Omega, \mathcal{A})$, with $T > 0$. We will use the following notations:

- $\| \,.\, \|$ is a norm on $\mathbb{R}^d$,
- $B_d(0,r)$ is d-ball with center 0 and radius r,
- $\mathrm{I\!I}$ is the indicator function,
- λ_n is the Lebesgue measure on $\mathbb{R}^n$.
- $dim_H S$ is the Hausdorff dimension of a set $S \subset \mathbb{R}^d$.
- $Im(X,\omega) = \{X_t(\omega); t \in [0,T]\}$ the image of X for $\omega \in \Omega$.

Definition: *For $r > 0$ one can define a $[0,1]$-valued random variables $C_T(r)$, and whenever it exists, the bivariate occupation measure dimension ν of X, such as:*

$$C_T(r) = \frac{1}{T^2}\int_0^T\int_0^T \mathrm{I\!I}_{(\|X_t-X_s\|\leq r)}dsdt. \tag{2.1}$$

$$\nu(\omega) = \lim_{r\to 0}\frac{\log C_T(r)}{\log r}. \tag{2.2}$$

The following property is obvious but useful:

Property 2.1 *For $\omega \in \Omega$, if $\| \,.\, \|_1$ and $\| \,.\, \|_2$ are two norms on $\mathbb{R}^d$, and whenever $\nu_1(\omega)$ (computed with $\| \,.\, \|_1$) exists, then $\nu_2(\omega)$ (computed with $\| \,.\, \|_2$) exists and $\nu_1(\omega) = \nu_2(\omega)$.*

Basic properties of bivariate occupation measure dimension can also be expressed from known results about occupation measures (we essentially use results of [1] and [9]). First, we give a relation between bivariate occupation measure and Hausdorff dimensions.

Property 2.2 *Whenever $\nu(\omega)$ exists for $\omega \in \Omega$, then*

$$\nu(\omega) \leq dim_H Im(X,\omega) \leq d. \tag{2.3}$$

Proof: Let us define an energy integral denoted $dim_C Im(X)$ such as:

$$dim_C Im(X,\omega) = \sup\left\{\alpha \in \mathbb{R}_+,\ \int_0^T\int_0^T \frac{dsdt}{\parallel X_t(\omega)-X_s(\omega)\parallel^\alpha} < +\infty\right\}. \qquad (2.4)$$

For a compact set in $\mathbb{R}^d$, capacity and Hausdorff dimensions are equal (see [8] and [11]). Now, it is obvious from capacity dimension definition that $dim_C Im(X,\omega)$ is smaller than or equal to capacity dimension of $Im(X,\omega)$. If $\nu(\omega)$ exists then it will be enough to show $dim_C Im(X,\omega) = \nu(\omega)$ for proving (2.3).
For a Borel set $J \in \mathbb{R}^+$, we define $\mu_Z(J,\omega)$ the occupation measure of $Z = \{Z_{s,t} = \parallel X_t - X_s \parallel, (s,t) \in [0,T]^2\}$ a $\mathbb{R}^+$-valued continuous random field on $[0,T]^2$. So $\mu_Z(J,\omega) = \lambda_2\{(s,t) \in [0,T]^2 / Z_{s,t}(\omega) \in J\}$. It is clear that:

$$C_T(r,\omega) = \frac{1}{T^2}\mu_Z([0,r],\omega). \qquad (2.5)$$

If $\nu(\omega)$ exists then $\mu_Z([0,r],\omega) = r^\nu l(r,\omega)$ with $\lim_{r\to 0}\frac{\log l(r,\omega)}{\log r} = 0$. Let α be a positive real number. Applying a usual result to the occupation measure $\mu_Z(.,\omega)$, one obtains:

$$\int_0^T\int_0^T \frac{dsdt}{\parallel X_t(\omega)-X_s(\omega)\parallel^\alpha} = \int_0^{R(\omega)} \frac{d\mu_Z(r,\omega)}{r^\alpha}, \qquad (2.6)$$

with $R(\omega) = \sup_{(s,t)\in[0,T]^2}(||X(t)-X(s)||) < +\infty$ *a.s.* because X is continuous. Thus:

$$\int_0^T\int_0^T \frac{dsdt}{\parallel X_t(\omega)-X_s(\omega)\parallel^\alpha} = \left[\frac{r^\nu l(r,\omega)}{r^\alpha}\right]_0^{R(\omega)} + \alpha\int_0^{R(\omega)} \frac{r^\nu l(r,\omega)}{r^{\alpha+1}}dr. \qquad (2.7)$$

Then, for $\alpha > \nu$ we have $\int_0^T\int_0^T \frac{dsdt}{\parallel X_t(\omega)-X_s(\omega)\parallel^\alpha} = +\infty$ and for $\alpha < \nu$ we have $\int_0^T\int_0^T \frac{dsdt}{\parallel X_t(\omega)-X_s(\omega)\parallel^\alpha} < +\infty$. As a consequence, $dim_C Im(X,\omega) = \nu(\omega)$ and (2.3) is proved. ■

Hausdorff and bivariate occupation measure dimensions can be relied to self-intersection local time. In fact, $C_T(r)$ can be seen as a self-intersection occupation measure of X. Let $Y = \{Y_{s,t} = X_t - X_s, (s,t) \in [0,T]^2\}$ be a $\mathbb{R}^d$-valued continuous random field on $[0,T]^2$.

Property 2.3 *Let V_0 be a 0-neighborhood of $\mathbb{R}^d$. If X has a self-intersection local time $\beta(x)$ for $x \in V_0$, i.e. Y has a local time for $x \in V_0$, which is continuous and positive for $x = 0$, then*

$$\nu \text{ exists and } \nu) = d = dim_H Im(X) \text{ for almost every } \omega \in \Omega. \qquad (2.8)$$

Proof: Define $\mu_Y(I)$ the occupation measure of Y in a Borel set $I \in \mathbb{R}^d$ for $(s,t) \in [0,T]^2$. So $\mu_Y(I) = \lambda_2\{(s,t) \in [0,T]^2 / Y_{s,t} \in I\}$. As a consequence, for any $r > 0$,

$$C_T(r) = \frac{1}{T^2}\mu_Y(B_d(0,r)). \qquad (2.9)$$

If X has a self-intersection local time $\beta(x)$ for $x \in V_0$, $\mu_Y(.)$ is absolutely continuous with respect to Lebesgue measure λ_d. We have:

$$\mu_Y(B_d(0,r)) = \int_{B_d(0,r)} \beta(x)dx.$$

From conditions on $\beta(.)$, for almost every $\omega \in \Omega$, we obtain for $r < R_0(\omega)$:

$$\log C_T(r,\omega) \sim \log r^d.$$

From Definition (2.2) one obtains (2.8). ■

It is also possible to give relations between local time and dimensions of X:

Property 2.4 *If X has a local time $\alpha(x)$ for $x \in \mathbb{R}^d$, and if $\alpha(.)$ is locally square integrable on $\mathbb{R}^d$,*

$$\text{then } \ \nu \text{ exists and } \nu = d = dim_H Im(X). \tag{2.10}$$

Proof: We know from Geman and Horowitz [9] that for almost every $\omega \in \Omega$:

$$\alpha(x,\omega) = \lim_{r\to 0} \frac{1}{r^d} \int_0^T \mathbb{I}_{(\|X_t(\omega)-x\| \leq r)} dt \ \text{ for every } x \in Im(X,\omega).$$

But, from the same trick than (2.6), we have:

$$\int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\| \leq r)} dtds \quad = \quad \int_0^T \int_{Im(X)} \mathbb{I}_{(\|X_t - x\| \leq r)} \alpha(x) dxdt,$$

with $Im(X)$ a compact set of $\mathbb{R}^d$, because X is a continuous process. Thus,

$$\begin{aligned} \lim_{r\to 0} \frac{1}{r^d} \int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\| \leq r)} dtds &= \int_{Im(X)} \alpha(x) \left(\lim_{r\to 0} \frac{1}{r^d} \int_0^T \mathbb{I}_{(\|X_t - x\| \leq r)} dt \right) dx \\ &= \int_{Im(X)} (\alpha(x))^2 dx, \end{aligned}$$

that exists under the condition on α. ■

Property 2.4 allows to bracket more precisely the value of ν.

Property 2.5 *Let $X_t = (X_t^{(1)}, .., X_t^{(d)})'$ and assume that it exists $k \in \mathbb{N}^*$ and $\{i_1, .., i_k\} \subset \{1, .., d\}$ such that $(X_t^{(i_1)}, .., X_t^{(i_k)})'$ has a locally square integrable local time $\alpha_k(x)$ for $x \in \mathbb{R}^k$. Whenever the bivariate occupation measure dimension ν of X exists, then*

$$1 \leq k \leq \nu \leq dim_H Im(X) \leq d. \tag{2.11}$$

Proof: Let $Z_t = (X_t^{(i_1)}, .., X_t^{(i_k)})'$ for $t \in [0,T]$. For all $(s,t) \in [0,T]^2$, if $\| X_t - X_s \| \leq r$ then $\| Z_t - Z_s \| \leq r$. Then,

$$C_T(r) = \frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\| \leq r)} dsdt \leq \frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|Z_t - Z_s\| \leq r)} dsdt = C_T^Z(r).$$

But from Property (2.4), $\lim_{r\to 0} \dfrac{\log C_T^Z(r)}{\log r} = k$. Thus, if ν exists, it verifies (2.11). ■

3 Index-α multidimensional processes

First, assume that X is a multidimensional fractional Brownian motions in $\mathbb{R}^d$ with index $(\alpha_1, \alpha_2, ..., \alpha_d)$, i.e. $X = \{X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)})', t \in [0,T]\}$ verifies the following assumptions:

- **(H1)** X is a continuous multidimensional Gaussian process in $\mathbb{R}^d$, with zero-mean and stationary increments.
- **(H2)** For each $i = 1,..,d$, there exist $\alpha_i \in]0,1[$ and $a_i > 0$ such that for all $t \in [0,T]$:

$$\mathbb{E}(X_t^{(i)} - X_0^{(i)})^2 = \sigma_i^2(t) = a_i |t|^{2\alpha_i}. \tag{3.1}$$

- **(H3)** For all $(s,t) \in [0,T]^2$, for each $(i,j) \in \{1,..,d\}^2$ such that $i \neq j$, $X_t^{(i)}$ is independent of $X_s^{(j)}$.

Then, the bivariate occupation measure of such a process verifies:

Proposition 3.1 *Let X verify assumptions* **(H1)**, **(H2)** *and* **(H3)**. *For $z = (z_1, ..., z_d) \in \mathbb{R}^d$, denote $\| z \|^2 = \dfrac{z_1^2}{2a_1} + \cdots + \dfrac{z_d^2}{2a_d}$ and assume $0 < \alpha_1 \leq .. \leq \alpha_d < 1$. Then,*

1. *if $\alpha_1 + .. + \alpha_d = 1$, for almost every $\omega \in \Omega$, there exists $r_0(\omega) > 0$, such that for all $0 < r < r_0(\omega)$,*

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|X_t(\omega) - X_s(\omega)\| \leq r)} ds dt \geq \left[\left(\frac{1-\alpha_d}{\alpha_d}\right)\frac{1}{T.\Gamma(1+d/2)}\right] r^d \log\frac{1}{r} \tag{3.2}$$

2. *if $\alpha_1 + .. + \alpha_d > 1$, let*
   - *$i_0 \in \{1, \cdots, d\}$ be such that $\alpha_1 + .. + \alpha_{i_0 - 1} < 1$ and $\alpha_1 + .. + \alpha_{i_0} \geq 1$,*
   - *$j_0 = \min\{j \in \{1, \cdots, i_0\}, \alpha_j = \alpha_{i_0}\}$ and $k_0 = \max\{k \in \{i_0, \cdots, d\}, \alpha_k = \alpha_{i_0}\}$*
   - *$\nu_0 = \dfrac{1 + \sum_{k=1}^{i_0}(\alpha_{i_0} - \alpha_k)}{\alpha_{i_0}}$.*

   Then,

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|X_t - X_s\| \leq r)} ds dt \underset{r\to 0}{\overset{a.s.}{\sim}} \left[\left(\frac{\alpha_{i_0+1} - \alpha_{i_0}}{\alpha_{i_0+1}\alpha_{i_0}}\right)\frac{1}{T.\Gamma(1+i_0/2)}\right] r^{i_0}\log\frac{1}{r} \tag{3.3}$$

if $\alpha_1 + \cdots + \alpha_{i_0} = 1$ and $\alpha_{i_0+1} \neq \alpha_{i_0}$;

$$\underset{r\to 0}{\overset{a.s.}{\sim}} \left[\frac{1}{\alpha_{i_0}}\left(\frac{1}{\nu_0 + 1 - j_0} + \frac{1}{k_0 - \nu_0}\right)\frac{1}{T.\Gamma(1+\nu_0/2)}\right] r^{\nu_0} \tag{3.4}$$

else.

By this way, we have two asymptotic behaviors of planar and spatial Brownian motion occupation measures:

Corollary 3.1 *If X is a $\mathbb{R}^d$-Brownian motion such that for all $(s,t) \in [0,T]^2$ and each $(i,j) \in \{1, \cdots, d\}$, $\mathbb{E}(X_t^{(i)})^2 = |t|$ and $X_t^{(i)}$ and $X_s^{(j)}$ are independent when $i \neq j$, and if $\| \,.\, \|_e$ is the usual Euclidean norm on $\mathbb{R}^d$,*

1. *if $d = 2$ (X is a planar Brownian motion), for almost every $\omega \in \Omega$, there exists $r_0(\omega) > 0$, such that for all $0 < r < r_0(\omega)$,*

$$\frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|X_t(\omega) - X_s(\omega)\|_e \le r)} ds dt \ge \frac{1}{4T} r^2 \log \frac{1}{r}. \tag{3.5}$$

2. *if $d \ge 3$ (X is a $\mathbb{R}^d$-spatial Brownian motion),*

$$\frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\|_e \le r)} ds dt \underset{r \to 0}{\overset{a.s.}{\sim}} \frac{d}{2(d-2)T} r^2. \tag{3.6}$$

Now we suppose that $X = \{X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)})', t \in [0,T]\}$ is a p-stable multidimensional symmetric process and verifies the following assumptions (we refer to Nolan, 1988, or Samorodnitsky and Taqqu, 1994, for details on p-stable symmetric processes):

- **(I1)** X is a continuous multidimensional symmetric p-stable process with stationary increments, values in $\mathbb{R}^d$ and $p = (p_1, p_2, \cdots, p_d) \in (0,2]^d$, i.e. for any $j = 1, \cdots, d$, $\lambda \in \mathbb{R}$, $t \in [0,T]$,
$$\mathbb{E} \exp\left(i\lambda X_t^{(j)}\right) = \exp\left(-|\lambda|^{p_j} \| X_t^{(j)} \|_{p_j}^{p_j}\right).$$

- **(I2)** For each $j = 1, .., d$, there exist $0 < \alpha_j \le \max(1, p_j^{-1})$ and $a_j > 0$ such that for all $(s,t) \in [0,T]^2$:
$$\sigma_j(t-s) = \| X_t^{(j)} - X_s^{(j)} \|_{p_j} = a_j |t-s|^{\alpha_j}.$$

- **(I3)** (Local nondeterministic assumption, LND) For each $j \in \{1, \cdots, d\}$,
  1. $\| X_t^{(j)} \|_{p_j} > 0$ for all $t \in [0,T]$.
  2. $\| X_t^{(j)} - X_s^{(j)} \|_{p_j} > 0$ for all $(s,t) \in [0,T]^2$ sufficiently close.
  3. For any $m > 1$, there is a $c_m > 0$, such that for all $(a_1, \cdots, a_m) \in \mathbb{R}^m$ and all $0 \le t_1 \le \cdots \le t_m \le T$:

$$\begin{aligned} c_m^{-1} \left(\| a_1 X_{t_1}^{(j)} \|_{p_j} + \sum_{l=2}^m \| a_l (X_{t_l}^{(j)} - X_{t_{l-1}}^{(j)}) \|_{p_j} \right) & \\ \le \| a_1 X_{t_1}^{(j)} + \sum_{l=2}^m a_l (X_{t_l}^{(j)} - X_{t_{l-1}}^{(j)}) \|_{p_j} & \\ \le c_m \left(\| a_1 X_{t_1}^{(j)} \|_{p_j} + \sum_{l=2}^m \| a_l (X_{t_l}^{(j)} - X_{t_{l-1}}^{(j)}) \|_{p_j} \right). & \end{aligned}$$

- **(I4)** For all $(s,t) \in [0,T]^2$, for each $(i,j) \in \{1,..,d\}^2$ such that $i \neq j$, $X_t^{(i)}$ is independent of $X_s^{(j)}$.
- **(I5)** (Local decorrelation assumption) There exist $0 < \gamma \leq 1$, $\delta > 0$, $\kappa > 0$ and $r_0 > 0$ such that for each $j \in \{1, \cdots, d\}$, for all $(\lambda, \lambda') \in \mathbb{R}_+^2$, for all $(s,t,s',t') \in [0,T]^4$ such that $\lambda|t-s| \leq r$, $\lambda'|t'-s'| \leq r$ and $|s-s'| \geq r^{1-\gamma}$ with $0 < r \leq r_0$:

$$\| \lambda(X_t^{(j)} - X_s^{(j)}) + \lambda'(X_{t'}^{(j)} - X_{s'}^{(j)}) \|_{p_j}^{p_j} \geq \left(\| \lambda_j(X_t^{(j)} - X_s^{(j)}) \|_{p_j}^{p_j} + \| \lambda_j'(X_{t'}^{(j)} - X_{s'}^{(j)}) \|_{p_j}^{p_j}\right)\left(1 - \kappa \frac{1}{r^\delta}\right).$$

Remark 1 : Several examples of such processes can be found in Nolan (1988), or Samorodnitsky and Taqqu (1994). The only non-classical assumption is the Local decorrelation assumption **(I5)**, but it is verified for instance by stable L/'evy or linear fractional stable motions (and then fractional Brownian motions). All these processes are self-similar processes and the self-similarity is peharps the only necessary property.

Remark 2 : Observe also that the case $p_i = 2$ for all $i = 1, \cdots, d$ corresponds to the Gaussian case, and in such a case, $\| X_t^{(j)} - X_s^{(j)} \|_{p_j}^{p_j} = 2\,\mathrm{var}(X_t^{(j)} - X_s^{(j)})$.

Proposition 3.2 *Let X verify assumptions* **(I1-5)**. *Assume that $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$ and for each $n \in \mathbb{N}^*$, denote $\| y \|_{\max} = \max_{i \in \{1,\cdots,n\}} (|y_i|)$ for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. Then,*

1. *if $\alpha_1 + .. + \alpha_d = 1$, for almost every $\omega \in \Omega$, there exists $r_0(\omega) > 0$, such that for all $0 < r < r_0(\omega)$,*

$$\frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|X_t(\omega) - X_s(\omega)\|_{\max} \leq r)} ds dt \geq \left[\frac{2}{\alpha_{i_0} T} \left(\frac{2}{\pi}\right)^d \prod_{k=1}^d \frac{1}{a_k p_k} \Gamma(\frac{1}{p_k})\right] r^d \log \frac{1}{r}; \quad (3.7)$$

2. *if $\alpha_1 + .. + \alpha_d > 1$, let*
   - *$i_0 \in \{1, \cdots, d\}$ be such that $\alpha_1 + .. + \alpha_{i_0 - 1} < 1$ and $\alpha_1 + .. + \alpha_{i_0} \geq 1$,*
   - *$j_0 = \min\{j \in \{1, \cdots, i_0\}, \alpha_j = \alpha_{i_0}\}$ and $k_0 = \max\{k \in \{i_0, \cdots, d\}, \alpha_k = \alpha_{i_0}\}$*
   - *$\nu_0 = \dfrac{1 + \sum_{k=1}^{i_0} (\alpha_{i_0} - \alpha_k)}{\alpha_{i_0}}$.*

   Then,

$$\frac{1}{T^2} \int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\|_{max} \leq r)} ds dt \overset{a.s.}{\underset{r\to 0}{\sim}}$$

$$\overset{a.s.}{\underset{r\to 0}{\sim}} \left[\frac{2}{T}\left(\frac{2}{\pi}\right)^{i_0} \left(\frac{1}{\alpha_{i_0}} - \frac{1}{\alpha_{i_0+1}}\right) \prod_{k=1}^{i_0} \frac{1}{a_k p_k} \Gamma(\frac{1}{p_k})\right] r^{\nu_0} \log \frac{1}{r}$$

$$\text{if } \alpha_1 + \cdots + \alpha_{i_0} = 1 \text{ and } \alpha_{i_0+1} \neq \alpha_{i_0}; \quad (3.8)$$

$$\overset{a.s.}{\underset{r\to 0}{\sim}} \left[\frac{2}{T}\left(\frac{2}{\pi}\right)^{k_0} \left(\prod_{k=1}^{j_0-1} \frac{1}{a_k p_k} \Gamma(\frac{1}{p_k})\right) \int_0^\infty \frac{dv}{v^{\alpha_1 + .. + \alpha_{j_0 - 1}}} \prod_{j=j_0}^{k_0} \int_0^\infty \frac{\sin \lambda}{\lambda} e^{-(a_j \lambda v^{\alpha_j})^{p_i}} d\lambda\right] r^{\nu_0}$$

$$\text{else.} \quad (3.9)$$

The particular case of isotropic stable L/'evy motions provides the following results:

Corollary 3.2 *Let X be a d-multidimensional p-stable L/'evy motion,* i.e. *X is a p-stable process having stationary independent increments such that $\mathbb{E}e^{i\theta X_t^{(i)}} = e^{-t|\theta|^p}$ for each $i = 1,..,d$ and $X^{(i)}$ and $X^{(j)}$ are independent for $i \neq j$. Then, if $p = 2$ and $d \geq 3$ (case of the spatial Brownian motion) or if $0 < p < 2$ and $d \geq 2$,*

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|X_t-X_s\|_{max}\leq r)}dsdt \underset{r\to 0}{\overset{a.s.}{\sim}} \frac{2}{T}\left(\frac{2}{\pi}\right)^d\left[\int_0^\infty dv\left(\int_0^\infty \frac{\sin\lambda}{\lambda}e^{-v\lambda^p}d\lambda\right)^d\right]r^p. \quad (3.10)$$

Finally these results implies the existence and value of the bivariate occupation measure dimension of these processes and more generally, of index-α multidimensional stable processes introduced by Nolan in [14] (in the Gaussian case, these processes were introduced by Adler in [1]). First, we add the following weaker assumptions :

- **(I2')** For $j = 1,\cdots,d$, set $\sigma_j(t) = \| X_t^{(j)} - X_0^{(j)} \|_{p_j}$. For each $j = 1,..,d$ there exists $\alpha_j \in (0, p_j]$ such that,

$$\alpha_j = \sup\{\alpha > 0,\ \sigma_j(t) = o(t^\alpha) \text{ for } t \to 0\} = \inf\{\alpha > 0,\ t^\alpha = o(\sigma_j(t)) \text{ for } t \to 0\}.$$

- **(I4')** (Locally approximatively independently components) For all $m \geq 1$, there is a $c = c(d,m) > 0$, such that for all $u_1,\cdots,u_m \in \mathbb{R}^d$ and all $t_1,\cdots,t_m \in [0,T]$,

$$\prod_{j=1}^d \mathbb{E}\exp\left(ic^{-1}\sum_{l=1}^m u_l^{(j)}X_{t_l}^{(j)}\right) \leq \mathbb{E}\exp\left(i\sum_{l=1}^m < u_l, X_{t_l} >\right) \leq \prod_{j=1}^d \mathbb{E}\exp\left(ic\sum_{l=1}^m u_l^{(j)}X_{t_l}^{(j)}\right).$$

Theorem 3.1 *If X verifies assumptions* **(I1), (I2'), (I3)** *and* **(I4')** *(*i.e. *X is an index-α stable process), for almost every $\omega \in \Omega$, the bivariate occupation measure dimension ν exists and*

$$\nu = dim_H Im(X) = \min\left(d, \min_{i\in\{1,..,d\}}\left\{\frac{1+\sum_{k=1}^i(\alpha_i-\alpha_k)}{\alpha_i}\right\}\right). \quad (3.11)$$

Then, we can identify the bivariate occupation measure dimension of a index-α stable process and its Hausdorff dimension (we use the expression of the Hausdorff dimension of this process image given in Xiao [22], who corrected a first result of Cuzick [4]). An advantage of the bivariate occupation measure dimension is its connection to the existence of the local time (see Pitt [16] and Nolan [15]) of such a process and thus the possibility of an explanation of the value of these dimensions:

- If $\sum_{k=1}^d \alpha_k < 1$, X is enough erratic to have a local time in $\mathbb{R}^d$ and thus $Im(X)$ "fills" his space.

- If $\sum_{k=1}^{d} \alpha_k > 1$, there is $i_0 \in \{2,..,d\}$ such as $\nu = \frac{1+\sum_{k=1}^{i_0}(\alpha_{i_0}-\alpha_k)}{\alpha_{i_0}}$ and ν verifies $\nu \in]i_0-1, i_0]$. So, ν is only depending on the i_0 most irregular X coordinates. Moreover, one notices that $\sum_{k=1}^{i_0-1} \alpha_k < 1$ and $\sum_{k=1}^{i_0} \alpha_k \geq 1$. If one considers the (i_0-1) most irregular X coordinates it exists a local time for their trajectory (see Proof of Theorem 3.1). By inequality (2.11) it is only natural that bivariate occupation measure dimension is greater than i_0-1. If one adds another coordinate then one can not define a local time for X and $\nu \leq i_0$.

4 Annex : Proofs

Set:
$$\begin{aligned} S(j) &= \alpha_1+\alpha_2+...+\alpha_j, \quad \text{and} \quad S(0)=0 \\ V(j) &= \frac{1+\sum_{k=1}^{j}(\alpha_j-\alpha_k)}{2\alpha_j}, \quad \text{and} \\ W(j) &= \frac{1+\sum_{k=1}^{j}(\alpha_j-\alpha_k)}{\alpha_j} \;=\; 2V(j). \end{aligned}$$

If $S(d) \geq 1$, let integer $i_0 \in \{2,...,d\}$ be such that $S(i_0-1) < 1$ and $S(i_0) \geq 1$.

Proof of Proposition 3.1: We turn to establish the asymptotic behavior of $C_T(r)$ when $r \to 0$ from the asymptotic behavior of occupation measure μ, where

$$\mu(x,I) = \lambda_2\{(s,t) \in I / \parallel X_t - X_s \parallel^2 \in [0,x]\}$$

for all $x \in \mathbb{R}_+$ and all Borel set $I \in [0,T]^2$. We have:

$$C_T(r) = \frac{1}{T^2}\mu(r^2,[0,T]^2).$$

If f is a measurable function on $\mathbb{R}_+$ then $\int_0^{+\infty} f(x)d\mu(x,I) = \int_I f(\parallel X_t - X_s \parallel^2)dsdt$. Let $f(x) = e^{-px}$ and denote $\hat{\mu}(p)$ the Laplace transform of $\mu(.,[0,T]^2)$. Thus,

$$\hat{\mu}(p) = \int_0^T \int_0^T \exp(-p \parallel X_t - X_s \parallel^2)dsdt. \tag{4.1}$$

The behavior of $\hat{\mu}(p)$ when $p \to +\infty$ is easier to be obtained than the behavior of $C_T(r)$ when $r \to 0$ which is deduced from a Tauberian theorem. By this way, we show the following lemmas:

Lemma 4.1 *Assume $S(d) > 1$. For large enough p, there exist positive real numbers $m_0(\alpha_1,\cdots,\alpha_d,T)$ and $d_1(\alpha_1,\cdots,\alpha_d,T)$ such that*

$$\mathbb{E}\hat{\mu}(p) = \; m_0\frac{\log p}{p^{V(i_0)}} + \mathcal{O}\left(\frac{\log p}{p^{V(i_0)+d_1}}\right) \quad \textit{if } S(i_0)=1 \textit{ and } \alpha_{i_0+1} \neq \alpha_{i_0}; \tag{4.2}$$

$$\mathbb{E}\hat{\mu}(p) = \; m_0\frac{1}{p^{V(i_0)}} + \mathcal{O}\left(\frac{1}{p^{V(i_0)+d_1}}\right) \quad \textit{else}, \tag{4.3}$$

$$\text{where } m_0(\alpha_1,\cdots,\alpha_d,T)=\begin{cases} T\left(\dfrac{\alpha_{i_0+1}-\alpha_{i_0}}{2\alpha_{i_0+1}\alpha_{i_0}}\right) & \text{if } S(i_0)=1 \text{ and } \alpha_{i_0+1}\neq\alpha_{i_0};\\ T\left(\dfrac{1}{1-S(j_0-1)}+\dfrac{1}{S(k_0)-1}\right) & \text{else;}\end{cases}$$
with $j_0=\min\{j\in\{1,\cdots,i_0\},\alpha_j=\alpha_{i_0}\}$ *and* $k_0=\max\{k\in\{i_0,\cdots,d\},\alpha_k=\alpha_{i_0}\}$.

Proof: We have:

$$\begin{aligned}\mathbb{E}\hat{\mu}(p) &= \int_0^T\int_0^T \mathbb{E}e^{-p\|X_t-X_s\|^2}dsdt\\ &= \int_0^T\int_0^T\prod_{i=1}^d \mathbb{E}\left[\exp\left(-\frac{p(X_t^i-X_s^i)^2}{2a_i}\right)\right]dsdt \quad \text{(independence)}\\ &= \int_0^T\int_0^T \frac{1}{\prod_{i=1}^d(1+p\mid t-s\mid^{2\alpha_i})^{1/2}}dsdt\\ &= 2\int_0^T\frac{T-u}{\prod_{i=1}^d(1+pu^{2\alpha_i})^{1/2}}du. \end{aligned}\tag{4.4}$$

Let's give an asymptotic development of (4.4).
We know that: $(1+x)^{-1/2}=\sum_{k=0}^{+\infty}a_k(x)=\sum_{k=0}^{+\infty}\frac{(2k)!}{4^k(k!)^2}(-x)^k$ if $x\in[0,1]$ and this power series uniformly converges on $[0,1]$.
Denote $\int_0^T\frac{T-u}{\prod_{i=1}^d(1+pu^{2\alpha_i})^{1/2}}du=\sum_{k=0}^d B_k(p)$, with

$$\begin{aligned}B_k(p) &= \int_{p^{-1/2\alpha_k}}^{p^{-1/2\alpha_{k+1}}}\frac{T-u}{\prod_{i=1}^d(1+pu^{2\alpha_i})^{1/2}}du \quad \text{for } k\in\{0,..,d-1\},\\ B_d(p) &= \int_{p^{-1/2\alpha_d}}^{T}\frac{T-u}{\prod_{i=1}^d(1+pu^{2\alpha_i})^{1/2}}du\end{aligned}$$

(with $1/\alpha_0=+\infty$). If $k<d$ and $u\in[p^{-1/2\alpha_k},p^{-1/2\alpha_{k+1}}]$ then $pu^{2\alpha_i}\geq 1$ for $i\in\{0,...,k\}$, and $pu^{2\alpha_i}\leq 1$ for $i\in\{k+1,...,d\}$. Thus:

$$\begin{aligned}B_k(p) &= \int_{p^{-1/2\alpha_k}}^{p^{-1/2\alpha_{k+1}}}\frac{(T-u)}{p^{k/2}u^{S(k)}}\prod_{i=1}^k(1+p^{-1}u^{-2\alpha_i})^{-1/2}\prod_{i=k+1}^d(1+pu^{2\alpha_i})^{-1/2}du\\ &= \int_{p^{-1/2\alpha_k}}^{p^{-1/2\alpha_{k+1}}}\frac{(T-u)}{p^{k/2}u^{S(k)}}\prod_{i=1}^k\left(\sum_{l_i=0}^{+\infty}a_{l_i}(p^{-1}u^{-2\alpha_i})\right)\prod_{i=k+1}^d\left(\sum_{l_i=0}^{+\infty}a_{l_i}(pu^{2\alpha_i})\right)du.\end{aligned}$$

Each series uniformly converges on $[p^{-1/2\alpha_k},p^{-1/2\alpha_{k+1}}]$ and their products uniformly converges on $[p^{-1/2\alpha_k},p^{-1/2\alpha_{k+1}}]$. Sums and integrals can be reversed:

$$B_k(p)=\sum_{l_1=0}^{+\infty}...\sum_{l_d=0}^{+\infty}\int_{p^{-1/2\alpha_k}}^{p^{-1/2\alpha_{k+1}}}\frac{(T-u)}{p^{k/2}u^{S(k)}}\prod_{i=1}^k a_{l_i}(p^{-1}u^{-2\alpha_i})\prod_{i=k+1}^d a_{l_i}(pu^{2\alpha_i})du.$$

But, after the integration, the main term in p of the sum is obtained for $l_1 = \cdots = l_d = 0$. As a consequence, for $k \in \{0, 1, .., d-1\}$, $B_k(p)$ has the following asymptotic development when p is large enough:

$$\text{-if } \alpha_k = \alpha_{k+1},\ B_k(p) = 0, \tag{4.5}$$

$$\text{-if } \alpha_k \neq \alpha_{k+1},\ B_k(p) = \left(\frac{T(\alpha_{k+1} - \alpha_k)}{2\alpha_{k+1}\alpha_k}\right) \frac{\log p}{p^{k/2}} + \mathcal{O}\left(\frac{\log p}{p^{k/2+d_k}}\right) \text{ if } S(k) = 1, \tag{4.6}$$

$$B_k(p) = \left(\frac{T}{S(k) - 1}\right) \frac{1}{p^{V(k)}} + \mathcal{O}\left(\frac{\log p}{p^{V(k)+d_k}}\right) \text{ if } S(k) > 1, \tag{4.7}$$

$$B_k(p) = \left(\frac{T}{1 - S(k)}\right) \frac{1}{p^{V(k+1)}} + \mathcal{O}\left(\frac{\log p}{p^{V(k+1)+d_k}}\right) \text{ if } S(k) < 1, \tag{4.8}$$

where $d_k(\alpha_1, ..., \alpha_k)$ is a positive real number.

We proceed similarly for $B_d(p)$. If $u \in [p^{-1/2\alpha_d}, T]$ then $pu^{2\alpha_i} \geq 1$ for all $i \in \{1, 2, ..., d\}$ and then when p is large enough:

$$\begin{aligned} B_d(p) &= \int_{p^{-1/2\alpha_d}}^{T} \frac{(T-u)}{p^{d/2} u^{S(d)}} \prod_{i=1}^{d} \left(\sum_{l_i=0}^{+\infty} a_{l_i} (p^{-1} u^{-2\alpha_i}) \right) du \\ &= \left(\frac{T}{S(d) - 1}\right) \frac{1}{p^{V(d)}} + \mathcal{O}\left(\frac{\log p}{p^{V(d)+d_k}}\right), \end{aligned} \tag{4.9}$$

where $d_d(\alpha_1, ..., \alpha_d)$ is a positive real number.

If $S(d) > 1$ then $V(i_0) = \inf\{V(1), ..., V(d)\}$ because i_0 verifies $S(i_0 - 1) < 1$ and $S(i_0) \geq 1$. Using (4.5), (4.6), (4.7), (4.8) and (4.9), one deduces that the main term of $B(p)$ asymptotic development is $\log p . p^{-V(i_0)}$ if $S(i_0) = 1$ and $\alpha_{i_0} \neq \alpha_{i_0+1}$, or $p^{-V(i_0)}$ else. After computations, it provides asymptotic developments (4.2) and (4.3). ■

From (4.2) and (4.3), it is reasonable to introduce the $\mathbb{R}^+$-valued random variable $m(p)$ such that:

- $m(p) = \dfrac{\hat{\mu}(p)\, p^{V(i_0)}}{\log p}$ when $S(i_0) = 1$ and $\alpha_{i_0+1} \neq \alpha_{i_0}$,
- $m(p) = \hat{\mu}(p)\, p^{V(i_0)}$ else.

Lemma 4.1 implies $\mathbb{E} m(p) \underset{p \to +\infty}{\longrightarrow} m_0(\alpha_1, ..., \alpha_d, T)$. We now show quadratic mean convergence of $m(p)$:

Lemma 4.2 *Assume $S(d) > 1$. There exist two positive real numbers $\zeta(\alpha_1, ..., \alpha_d, T)$ and $M(\alpha_1, ..., \alpha_d, T)$, such that for large enough p:*

$$\mathbb{E}(m(p) - m_0)^2 \leq \frac{M}{p^{2\zeta}}$$

Proof: We have:

$$\mathbb{E}(\hat{\mu}(p))^2 = \mathbb{E}\left[\int_0^T \int_0^T e^{-p\|X_t - X_s\|^2} dsdt\right]^2$$

$$= \int_0^T \int_0^T \int_0^T \int_0^T \prod_{i=1}^d \mathbb{E}\left[\exp\left(-\frac{p((X_t^i - X_s^i)^2 + (X_{t'}^i - X_{s'}^i)^2)}{2a_i}\right)\right] dsdtds'dt'$$

Letting $W^i_{s,t,s',t'} = \frac{1}{\sqrt{2a_i}}\begin{pmatrix} X_t^i - X_s^i \\ X_{t'}^i - X_{s'}^i \end{pmatrix}$, then:

$$\mathbb{E}\left[\exp\left(-\frac{p((X_t^i - X_s^i)^2 + (X_{t'}^i - X_{s'}^i)^2)}{2a_i}\right)\right] = \frac{1}{\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'}))^{1/2}}$$

(det is determinant, I_2 the identity matrix of $\mathbb{R}^2$ and cov the covariance matrix). Kono showed in proof of Theorem 2 [12] the existence of $0 < \epsilon_i \leq 1$ such that, for all $(s, t, s', t') \in [0, T]^4$:

$$\det(\mathrm{cov}(W^i_{s,t,s',t'})) \geq \epsilon_i^2 \mathbb{E}(X_t^i - X_s^i)^2 \mathbb{E}(X_{t'}^i - X_{s'}^i)^2.$$

Thus, $\forall (s, t, s', t') \in [0, T]^4$,

$$\begin{aligned}\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'})) &\geq 1 + p(|t-s|^{2\alpha_i} + |t'-s'|^{2\alpha_i}) + \epsilon_i^2(|t-s|^{2\alpha_i}|t'-s'|^{2\alpha_i}) \\ &\geq \epsilon_i^2(1 + p(|t-s|^{2\alpha_i})(1 + p(|t'-s'|^{2\alpha_i}). \end{aligned} \tag{4.10}$$

Moreover, $\forall (s, t, s', t') \in [0, T]^4$,

$$\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'})) = 1 + p(|t-s|^{2\alpha_i} + |t'-s'|^{2\alpha_i}) + ..$$
$$+ p^2(|t-s|^{2\alpha_i}|t'-s'|^{2\alpha_i} - \frac{1}{4}(|t-s'|^{2\alpha_i} + |t'-s|^{2\alpha_i} - |t-t'|^{2\alpha_i} - |s-s'|^{2\alpha_i})^2).$$

If $P(u, v, w) = (|w+u|^{2\alpha_i} + |w-v|^{2\alpha_i} - |u-v+w|^{2\alpha_i} - |w|^{2\alpha_i})^2$ then, for $|w| \geq p^{-1/4}$, $|u| \leq p^{-1/2}$ and $|v| \leq p^{-1/2}$, and large enough p,

$$\begin{aligned} P(u, v, w) &= |w|^{4\alpha_i}(|1 + \frac{u}{w}|^{2\alpha_i} + |1 - \frac{v}{w}|^{2\alpha_i} - |1 + \frac{u-v}{w}|^{2\alpha_i} - 1)^2 \\ &= 4\alpha_i^2(2\alpha_i - 1)^2|w|^{4\alpha_i - 4}\left(4u^2v^2 + O(\frac{u^3}{w}) + O(\frac{v^3}{w})\right) \\ &\leq 4\eta_i p^{1-\alpha_i}u^2v^2, \end{aligned}$$

with $0 < \eta_i \leq 8\alpha_i^2(2\alpha_i - 1)^2$ for large enough p. This implies:

$$\begin{aligned} |u|^{2\alpha_i}|v|^{2\alpha_i} - \frac{1}{4}P(u, v, w) &\geq |u|^{2\alpha_i}|v|^{2\alpha_i}(1 - \eta_i p^{1-\alpha_i}|u|^{2\alpha_i - 2}|v|^{2\alpha_i - 2}) \\ &\geq (1 - \eta_i p^{\alpha_i - 1})|u|^{2\alpha_i}|v|^{2\alpha_i}. \end{aligned}$$

Thus, if $|t-s| \leq p^{-1/2}$, $|t'-s'| \leq p^{-1/2}$ and $|s-s'| \geq p^{-1/4}$, when p is large enough,

$$\begin{aligned} \det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'})) &\geq 1 + p(|t-s|^{2\alpha_i} + |t'-s'|^{2\alpha_i}) + p^2(1 - \eta_i p^{\alpha_i - 1})|t-s|^{2\alpha_i}|t'-s'|^{2\alpha_i} \\ &\geq (1 - \eta_i p^{\alpha_i - 1})(1 + p(|t-s|^{2\alpha_i})(1 + p|t'-s'|^{2\alpha_i}). \end{aligned} \tag{4.11}$$

Thus, $\mathbb{E}(\hat{\mu}(p))^2 = \int_0^T\int_0^T\int_0^T\int_0^T \prod_{i=1}^d \frac{1}{\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'}))^{1/2}} ds dt ds' dt'$

$$= \int_{[0,T]^4} \mathbb{I}_{|t-s|\le p^{-1/2}}\mathbb{I}_{|t'-s'|\le p^{-1/2}}\mathbb{I}_{|s-s'|\ge p^{-1/4}} \prod_{i=1}^d \frac{1}{\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'}))^{1/2}} ds dt ds' dt' + ..$$

$$+ \int_{[0,T]^4} (1 - \mathbb{I}_{|t-s|\le p^{-1/2}}\mathbb{I}_{|t'-s'|\le p^{-1/2}}\mathbb{I}_{|s-s'|\ge p^{-1/4}}) \prod_{i=1}^d \frac{1}{\det(I_2 + p\mathrm{cov}(W^i_{s,t,s',t'}))^{1/2}} ds dt ds' dt'.$$

Denote I_1 and I_2 both previous integrals. With (4.11),

$$\begin{aligned} I_1 &\le \int_{[0,T]^4} \mathbb{I}_{|t-s|\le p^{-1/2}}\mathbb{I}_{|t'-s'|\le p^{-1/2}}\mathbb{I}_{|s-s'|\ge p^{-1/4}} \prod_{i=1}^d \frac{(1-\eta_i p^{\alpha_i-1})^{-1/2}}{(1+p(|t-s|^{2\alpha_i})^{1/2}(1+p|t'-s'|^{2\alpha_i})^{1/2}} \\ &\le 4\prod_{i=1}^d (1-\eta_i p^{\alpha_i-1})^{-1/2}\left(\int_0^{p^{-1/2}} \frac{T-u}{\prod_{i=1}^d (1+pu^{2\alpha_i})^{1/2}} du\right)^2 \\ &\le \prod_{i=1}^d (1-\eta_i p^{\alpha_i-1})^{-1/2} (\mathbb{E}\hat{\mu}(p))^2. \end{aligned} \tag{4.12}$$

As $1 - \mathbb{I}_{|t-s|\le p^{-1/2}}\mathbb{I}_{|t'-s'|\le p^{-1/2}}\mathbb{I}_{|s-s'|\ge p^{-1/4}} \le \mathbb{I}_{|t-s|\ge p^{-1/2}} + \mathbb{I}_{|t'-s'|\ge p^{-1/2}} + \mathbb{I}_{|s-s'|\le p^{-1/4}}$ and from (4.10), one obtains

$$\begin{aligned} I_2 &\le \int_{[0,T]^4} \mathbb{I}_{|t-s|\ge p^{-1/2}} \prod_{i=1}^d \frac{\epsilon_i^{-1}}{(1+p(|t-s|^{2\alpha_i})^{1/2}(1+p|t'-s'|^{2\alpha_i})^{1/2}} ds dt ds' dt' \\ &+ \int_{[0,T]^4} \mathbb{I}_{|t'-s'|\ge p^{-1/2}} \prod_{i=1}^d \frac{\epsilon_i^{-1}}{(1+p(|t-s|^{2\alpha_i})^{1/2}(1+p|t'-s'|^{2\alpha_i})^{1/2}} ds dt ds' dt' \\ &+ \int_{[0,T]^4} \mathbb{I}_{|s-s'|\le p^{-1/4}} \prod_{i=1}^d \frac{\epsilon_i^{-1}}{(1+p(|t-s|^{2\alpha_i})^{1/2}(1+p|t'-s'|^{2\alpha_i})^{1/2}} ds dt ds' dt' \\ &\le 2(\prod_{i=1}^d \epsilon_i^{-1})(\mathbb{E}\hat{\mu}(p)) \int_{p^{-1/2}}^T \frac{2(T-u)}{\prod_{i=1}^d (1+pu^{2\alpha_i})^{1/2}} du \\ &+ 4(\prod_{i=1}^d \epsilon_i^{-1}) \int_{[-T,T]^4} \mathbb{I}_{|w|\le p^{-1/4}} \frac{(T-u)(T-v)}{\prod_{i=1}^d (1+pu^{2\alpha_i})^{1/2}(1+pv^{2\alpha_i})^{1/2}} du dv dw dz \\ &\le 2(\prod_{i=1}^d \epsilon_i^{-1})(\mathbb{E}\hat{\mu}(p)) \frac{\log p}{p^{d/2}} + \frac{16T\prod_{i=1}^d \epsilon_i^{-1}}{p^{1/4}} (\mathbb{E}\hat{\mu}(p))^2 \end{aligned} \tag{4.13}$$

(one bounds $\int_{p^{-1/2}}^T \frac{(T-u)}{\prod_{i=1}^d (1+pu^{2\alpha_i})^{1/2}} du$ by the same computation as $B_d(p)$ in Lemma 4.1). Inequalities (4.12) and (4.13) imply that:

$$\mathbb{E}(\hat{\mu}(p))^2 \le (\mathbb{E}\hat{\mu}(p))^2 \left(\prod_{i=1}^d (1-\eta_i p^{\alpha_i-1})^{-1/2} + \frac{16T\prod_{i=1}^d \epsilon_i^{-1}}{p^{1/4}} + 2(\prod_{i=1}^d \epsilon_i^{-1}) \frac{\log p}{\mathbb{E}\hat{\mu}(p) p^{d/2}}\right)$$

As $S(d) > 1$, $\mathbb{E}\hat{\mu}(p) \sim \dfrac{m_0}{p^{V(i_0)}}$ and $V(i_0) < d$, thus $\dfrac{\log p}{\mathbb{E}\hat{\mu}(p)p^{d/2}} \leq \dfrac{1}{p^{(d/4-V(i_0)/4)}}$ for large enough p. So, there are two positive real numbers $\zeta_1(\alpha_1, ..., \alpha_d, T)$ and $M_1(\alpha_1, ..., \alpha_d, T)$ such that:

$$\begin{aligned} \mathbb{E}(\hat{\mu}(p))^2 - (\mathbb{E}\hat{\mu}(p))^2 &\leq \frac{M_1}{p^{2\zeta_1}}(\mathbb{E}\hat{\mu}(p))^2, \\ \mathbb{E}(m(p) - \mathbb{E}m(p))^2 &\leq \frac{M_1}{p^{2\zeta_1}}. \end{aligned} \tag{4.14}$$

Using Lemma 4.1, there are two positive real numbers $\zeta_2(\alpha_1, ..., \alpha_d, T)$ and $M_2(\alpha_1, ..., \alpha_d, T)$ such that

$$(\mathbb{E}m(p) - m_0)^2 \leq \frac{M_2}{p^{2\zeta_2}}. \tag{4.15}$$

By (4.14) and (4.15), one deduces Lemma 4.2. ■

Lemma 4.3 *Assume $S(d) > 1$. Then, $m(p) \underset{p\to+\infty}{\overset{a.s.}{\longrightarrow}} m_0$.*

Proof: From Lemma 4.2, denote $u_n = m(n^{1/\zeta})$ and thus $\mathbb{E}(u_n - \mathbb{E}u_n)^2 \leq \dfrac{M}{n^2}$. With Tchebychev Inequality, Borel-Cantelli Lemma and using $\mathbb{E}u_n \to m_0$ when $n \to +\infty$, one obtains:

$$u_n \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} m_0. \tag{4.16}$$

$(\hat{\mu}(p))_p$ is a decreasing sequence for all ω (because $\hat{\mu}(p,\omega) = \displaystyle\int_0^T \int_0^T e^{-p\|X_t(\omega)-X_s(\omega)\|^2} dsdt$ and $\| X_t(\omega) - X_s(\omega) \|^2$ is a $\mathbb{R}_+$-valued field). So, for $p \in [n^{1/\zeta}, (n+1)^{1/\zeta}]$ and for all ω,

$$\frac{n^{V(i_0)/\zeta}}{(\log(n+1)^{1/\zeta})^{e_0}}\hat{\mu}((n+1)^{1/\zeta},\omega) \leq \frac{p^{V(i_0)}}{(\log p)^{e_0}}\hat{\mu}(p,\omega) \leq \frac{(n+1)^{V(i_0)/\zeta}}{(\log n^{1/\zeta})^{e_0}}\hat{\mu}(n^{1/\zeta},\omega),$$

$$\left(\frac{n}{n+1}\right)^{V(i_0)/\zeta}\left(\frac{\log n}{\log(n+1)}\right)^{e_0} u_{n+1} \leq m(p) \leq \left(\frac{n+1}{n}\right)^{V(i_0)/\zeta}\left(\frac{\log(n+1)}{\log n}\right)^{e_0} u_n,$$

where $e_0 = 1$ if $S(i_0) = 1$ and $\alpha_{i_0} \neq \alpha_{i_0+1}$, else $e_0 = 0$. When $n \to +\infty$, with (4.16) it achieves the proof. ■

End of the proof of Proposition 3.1: • if $S(d) > 1$, it is now possible to prove (3.3) and (3.4):

One has just shown that $m(p) \underset{p\to+\infty}{\overset{a.s.}{\longrightarrow}} m_0$. So, for almost every ω,

$$\hat{\mu}(p,\omega) \sim \frac{m_0(\log p)^{e_0}}{p^{V(i_0)}} \text{ when } p \to +\infty, \tag{4.17}$$

where $e_0 = 1$ if $S(i_0) = 1$ and $\alpha_{i_0} \neq \alpha_{i_0+1}$, else $e_0 = 0$. For each ω verifying (4.17), one applies a Karamata Tauberian theorem to $\hat{\mu}(p,\omega)$. More precisely, we have conditions for applying

Theorem 3, chap. XIII, of Feller (1971) (the possible logarithm of (4.17) is a slow varying function). As a consequence, for almost every ω,

$$\mu([0,r],\omega) \sim \frac{m_0}{\Gamma(V(i_0)+1)}(\log\frac{1}{r})^{e_0} r^{V(i_0)} \text{ when } r \to 0, \tag{4.18}$$

$$\text{thus } C_T(r,\omega) \sim \frac{m_0}{\Gamma(V(i_0)+1)T^2}(2\log\frac{1}{r})^{e_0} r^{2V(i_0)} \text{ when } r \to 0. \tag{4.19}$$

• if $S(d)=1$, we cannot prove the almost sure convergence as in previous proof. In fact, we can show a $\mathbb{L}^2$-convergence of $m(p)$ to m_0, but the rate of convergence (less quick than $\frac{1}{\log p}$, see [2]) is not sufficient to obtain a a.s.-convergence. More, $\mathbb{L}^2$ or probability convergence is not kept by the Tauberian Theorem.
Nevertheless, we can obtain the inequality (3.2). Denote $Y=\{Y_t, t\in[0,T]\}$, where $Y_t = (X_t^{(1)},..,X_t^{(d)},X_t^{(d+1)})$ is a $\mathbb{R}^{d+1}$-values process, with $X_t^{(d+1)}$ a fractional Brownian motion with parameter $1>\alpha_{d+1}>\alpha_d$ such that $X_t^{(d+1)}$ is independent of $X_s^{(i)}$ for all $i=1,\cdots,d$ and $(s,t)\in[0,T]^2$. It is obvious that if $\| Y_t - Y_s \| \leq r$ then $\| X_t - X_s \| \leq r$. Then,

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|X_t-X_s\|\leq r)} dsdt \geq \frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|Y_t-Y_s\|\leq r)} dsdt.$$

From (4.19), it exists $C>0$, such that for $r\to 0$ and almost every ω,

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|Y_t-Y_s\|\leq r)} dsdt \sim C r^d \log(\frac{1}{r}),$$

because $S(d+1)>1$, $S(d)=1$ and $\alpha_{d+1}>\alpha_d$. Thus, for almost every $\omega\in\Omega$, it exists $r_0(\omega)>0$, such that for all $0<r\leq r_0(\omega)$,

$$\frac{1}{T^2}\int_0^T\int_0^T \mathbb{I}_{(\|X_t-X_s\|\leq r)} dsdt \geq \frac{C}{2} r^d \log(\frac{1}{r}). \ \blacksquare \tag{4.20}$$

Proof of Proposition 3.2:

Lemma 4.4 *Under hypothesis of Proposition 3.2, there exists a real number s_0 only depending on $(\alpha_i)_i$, $(a_i)_i$, $(p_i)_i$ and T, such that for small enough r,*

$$\mathbb{E}C_T(r) = s_0 r^{W(i_0)}\log\frac{1}{r} + \mathcal{O}\left(r^{W(i_0)+d_1}\right) \quad \textit{if } S(i_0)=1 \textit{ and } \alpha_{i_0+1}\neq\alpha_{i_0}; \tag{4.21}$$

$$\mathbb{E}C_T(r) = s_0 r^{W(i_0)} + \mathcal{O}\left(r^{W(i_0)+d_1}\right) \quad \textit{else,} \tag{4.22}$$

$$\textit{where } s_0 = \begin{cases} \bullet\ \dfrac{2^{2d+1}}{T}\left(\dfrac{\pi}{2}\right)^{d-i_0}\left(\dfrac{1}{\alpha_{i_0}}-\dfrac{1}{\alpha_{i_0+1}}\right)\displaystyle\prod_{k=1}^{i_0}\frac{1}{a_k p_k}\Gamma(\frac{1}{p_k}) & \textit{if } S(i_0)=1 \textit{ and } \alpha_{i_0+1}\neq\alpha_{i_0}, \\ \bullet\ \left(\displaystyle\int_0^\infty \frac{1}{v^{S(j_0-1)}}\left(\prod_{j=j_0}^{k_0}\int_0^\infty \frac{\sin\lambda}{\lambda} e^{-(a_j\lambda v^{\alpha_j})^{p_i}} d\lambda\right) dv\right)\times .. & \\ \qquad \times\dfrac{2^{2d+1}}{T}\left(\dfrac{\pi}{2}\right)^{d-k_0}\left(\displaystyle\prod_{k=1}^{j_0-1}\frac{1}{a_k p_k}\Gamma(\frac{1}{p_k})\right) & \textit{else.} \end{cases}$$

with $j_0=\min\{j\in\{1,\cdots,i_0\},\alpha_j=\alpha_{i_0}\}$ and $k_0=\max\{k\in\{i_0,\cdots,d\},\alpha_k=\alpha_{i_0}\}$.

Proof: From stable process definition and assumption **(H3')**, for $(s,t) \in [0,T]^2$ and with $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}^d$, we have:

$$\mathbb{E} \exp\left(i < \lambda, X_t - X_s >\right) = \prod_{j=1}^{d} \exp\left(-|\lambda_j|^{p_j} \parallel X_t^{(j)} - X_s^{(j)} \parallel_{p_j}^{p_j}\right),$$

because $X_t - X_s$ is also a p-stable process. As a consequence,

$$\begin{aligned}
\mathbb{E} C_T(r) &= \mathbb{E}\left(\frac{1}{T^2}\int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\|_{\max} \leq r)} ds dt\right) \\
&= \frac{1}{(2\pi)^d T^2}\int_0^T \int_0^T \int_{\mathbb{R}^d} \overline{\left(\widehat{\mathbb{I}_{(\|x\|_{\max} \leq r)}}\right)(\lambda)}\, \mathbb{E} \exp\left(i < \lambda, X_t - X_s >\right) d\lambda ds dt \text{ (Parceval)} \\
&= \frac{1}{(2\pi)^d T^2}\int_0^T \int_0^T \int_{\mathbb{R}^d} 2^d \prod_{j=1}^{d} \frac{\sin(r\lambda_j)}{\lambda_j} \exp\left(-|\lambda_j|^{p_j} \parallel X_t^{(j)} - X_s^{(j)} \parallel_{p_j}^{p_j}\right) d\lambda_1 \cdots d\lambda_d ds dt \\
&= 2\frac{2^d}{\pi^d T^2}\int_0^T (T-u) \prod_{j=1}^{d} F_{p_i}\left(\frac{r}{\sigma_j(u)}\right) du,
\end{aligned}$$

where $F_p(\varepsilon) = \displaystyle\int_0^\infty \frac{\sin \lambda}{\lambda} \exp\left(-\left(\frac{\lambda}{\varepsilon}\right)^p\right) d\lambda$. When $\varepsilon \to 0$, we have the two following expansions (with classical proofs) of the function F_p with $K(p) > 0$ depending only on p:

- $$\begin{aligned}\left|F(\varepsilon) - \frac{1}{p}\Gamma(\frac{1}{p})\varepsilon\right| &= \left|\int_0^\infty \frac{\sin(\varepsilon\lambda)}{\lambda} \exp\left(-|\lambda|^p\right) d\lambda - \varepsilon \int_0^\infty \exp\left(-\lambda^p\right) d\lambda\right| \\ &\leq K(p)\varepsilon^3 \end{aligned} \tag{4.23}$$

- $$\begin{aligned}\left|F(\frac{1}{\varepsilon}) - \frac{\pi}{2}\right| &= \left|\int_0^\infty \frac{\sin(\lambda)}{\lambda} \exp\left(-\left(\varepsilon\lambda\right)^p\right) d\lambda - \int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda\right| \\ &\leq K(p)\varepsilon^{p/2}. \end{aligned} \tag{4.24}$$

First assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_d$ and set $0 < \beta_1 < \alpha_1 < \beta_2 < \cdots < \alpha_d < \beta_{d+1}$.
If $r \to 0$, for $i = 1, \cdots, d-1$ and $u \in [r^{\beta_i^{-1}}, r^{\beta_{i+1}^{-1}}]$ then:

- for $j \leq i-1$, then $\dfrac{r}{\sigma_j(u)} \sim \dfrac{r}{a_j u^{\alpha_j}} \to 0$ and thus,

$$\left|F(\frac{r}{\sigma_j(u)}) - \frac{1}{p_j}\Gamma(\frac{1}{p_j})\frac{r}{\sigma_j(u)}\right| \leq K(p_j)\left(\frac{r}{\sigma_j(u)}\right)^3.$$

- if $j \geq i+1$, then $\dfrac{r}{\sigma_j(u)} \sim \dfrac{r}{a_j u^{\alpha_j}} \to \infty$ and thus,

$$\left|F(\frac{r}{\sigma_j(u)}) - \frac{\pi}{2}\right| \leq K(p_j)\left(\frac{r}{\sigma_j(u)}\right)^{p_j/2}.$$

Then, for $u \in [r^{\beta_i^{-1}}, r^{\beta_{i+1}^{-1}}]$ and $r \to 0$, it exists $c_i > 0$ such that

$$\prod_{j=1}^{d} \int_0^\infty \frac{\sin\lambda}{\lambda} \exp\left[-\lambda^{p_j}\left(\frac{\sigma_j(u)}{r}\right)^{p_j}\right] d\lambda \quad =$$

$$= \left(\frac{\pi}{2}\right)^{d-i}\left(\prod_{k=1}^{i-1}\frac{1}{a_k p_k u^{\alpha_k}}\Gamma(\frac{1}{p_k})\right) r^{i-1}(1+\mathcal{O}(r^{c_i}))\int_0^\infty \frac{\sin\lambda}{\lambda}\exp\left[-\lambda^{p_i}\left(\frac{\sigma_i(u)}{r}\right)^{p_i}\right] d\lambda,$$

and thus,

$$\int_{r^{\beta_i^{-1}}}^{r^{\beta_{i+1}^{-1}}} \prod_{j=1}^{d}\int_0^\infty \frac{\sin\lambda}{\lambda}\exp\left[-\lambda^{p_j}\left(\frac{\sigma_j(u)}{r}\right)^{p_j}\right] d\lambda du =$$

$$= \left(\frac{\pi}{2}\right)^{d-i}\left(\prod_{k=1}^{i-1}\frac{1}{a_k p_k}\Gamma(\frac{1}{p_k})\right) r^{i-1}\left(1+\mathcal{O}(r^{c_i})\right)\int_{r^{\beta_i^{-1}}}^{r^{\beta_{i+1}^{-1}}}\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{u^{S(i-1)}}\exp\left[-\left(\frac{a_i\lambda u^{\alpha_i}}{r}\right)^{p_i}\right] d\lambda du$$

$$= \left(\frac{\pi}{2}\right)^{d-i}\left(\prod_{k=1}^{i-1}\frac{1}{a_k p_k}\Gamma(\frac{1}{p_k})\right) r^{W(i)}\left(1+\mathcal{O}(r^{c_i})\right)\int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}}\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{v^{S(i-1)}} e^{-(a_i\lambda v^{\alpha_i})^{p_i}} d\lambda dv,$$

with $u = v r^{\alpha_i^{-1}}$. As a consequence, when $r \to 0$, we have 3 different cases:

- If $S(i-1) < 1$ then,

$$\int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}}\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{v^{S(i-1)}} e^{-(a_i\lambda v^{\alpha_i})^{p_i}} d\lambda dv =$$

$$= \begin{cases} \displaystyle\int_1^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}} \frac{1}{a_i p_i}\Gamma(\frac{1}{p_i})\frac{1}{v^{S(i)}} dv + \mathcal{O}(1) \\ \displaystyle\qquad = \frac{\left(1+\mathcal{O}(r^{d_i})\right)}{a_i p_i (1-S(i))}\Gamma(\frac{1}{p_i}) r^{(1-S(i))(\beta_{i+1}^{-1}-\alpha_i^{-1})}, & \text{if } S(i) < 1, \\ \displaystyle\int_1^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}} \frac{1}{a_i p_i}\Gamma(\frac{1}{p_i})\frac{1}{v^{S(i)}} dv + \mathcal{O}(1) \\ \displaystyle\qquad = \frac{1}{a_i p_i}\Gamma(\frac{1}{p_i})\left(\frac{1}{\beta_{i+1}} - \frac{1}{\alpha_i}\right)\log r + \mathcal{O}(1), & \text{if } S(i) = 1, \\ \displaystyle\int_0^\infty\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{v^{S(i-1)}} e^{-(a_i\lambda v^{\alpha_i})^{p_i}} d\lambda dv \left(1+\mathcal{O}(r^{d_i})\right) \\ \displaystyle\qquad = \frac{\left(1+\mathcal{O}(r^{d_i})\right)}{p_i\, a_i^{(1-S(i-1))\alpha_i^{-1}}}\Gamma(\frac{1-S(i-1)}{\alpha_i p_i})\int_0^\infty \frac{\sin\lambda}{\lambda^{2-(S(i)-1)\alpha_i^{-1}}} d\lambda, & \text{if } S(i) > 1. \end{cases}$$

- If $S(i-1)=1$ then,

$$\int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}}\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{v}e^{-(a_i\lambda v^{\alpha_i})^{p_i}}d\lambda dv \quad = \quad \int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{1}\frac{\pi}{2}\frac{dv}{v}+\mathcal{O}(1)$$
$$= \quad \frac{\pi}{2}\left(\frac{1}{\beta_i}-\frac{1}{\alpha_i}\right)\log\frac{1}{r}+\mathcal{O}(1).$$

- If $S(i-1)>1$ then,

$$\int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{r^{\beta_{i+1}^{-1}-\alpha_i^{-1}}}\int_0^\infty \frac{\sin\lambda}{\lambda}\frac{1}{v^{S(i-1)}}e^{-(a_i\lambda v^{\alpha_i})^{p_i}}d\lambda dv \quad = \quad \int_{r^{\beta_i^{-1}-\alpha_i^{-1}}}^{1}\frac{\pi}{2}\frac{dv}{v^{S(i-1)}}+\mathcal{O}(1)$$
$$= \quad \frac{\pi}{2}\frac{r^{(1-S(i-1))(\beta_i^{-1}-\alpha_i^{-1})}}{(S(i-1)-1)}\left(1++\mathcal{O}(r^{d_i})\right).$$

More, as $S(d)>1$, we have:

$$\int_{r^{\beta_{d+1}^{-1}}}^{T}(T-u)\prod_{j=1}^{d}\int_0^\infty\frac{\sin\lambda}{\lambda}\exp\left[-\lambda^{p_j}\left(\frac{\sigma_j(u)}{r}\right)^{p_j}\right]d\lambda du =$$
$$= \left(\prod_{k=1}^{d}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\right)r^d\left(\int_{r^{\beta_{d+1}^{-1}}}^{T}(T-u)\frac{du}{u^{S(d)}}\right)(1+\mathcal{O}(r^{c_{d+1}}))$$
$$= \left(\prod_{k=1}^{d}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\right)\frac{1}{(S(d)-1)}r^dr^{(1-S(d))\beta_{d+1}^{-1}}(1+\mathcal{O}(r^{c_{d+1}})),$$

and

$$\int_0^{r^{\beta_1^{-1}}}\prod_{j=1}^{d}\int_0^\infty\frac{\sin\lambda}{\lambda}\exp\left[-\lambda^{p_j}\left(\frac{\sigma_j(u)}{r}\right)^{p_j}\right]d\lambda du=\left(\frac{\pi}{2}\right)^d r^{\beta_1^{-1}}(1+\mathcal{O}(r^{c_0})).$$

As a conclusion, if $0<\alpha_1<\cdots<\alpha_d$, when $r\to 0$,

- if $S(i_0-1)<1$ and $S(i_0)>1$ then

$$\mathbb{E}C_T(r)=\left[\frac{2}{T}\left(\frac{2}{\pi}\right)^{i_0}\frac{1}{p_{i_0}a_{i_0}^{(1-S(i_0-1)\alpha_{i_0}^{-1}}}\Gamma(\frac{1-S(i_0-1)}{\alpha_{i_0}p_{i_0}})\prod_{k=1}^{i_0-1}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\times ..\right.$$
$$\left... \times\left(\int_0^\infty\frac{\sin\lambda}{\lambda^{2-(S(i_0)-1)\alpha_{i_0}^{-1}}}d\lambda\right)\right]r^{W(i_0)}(1+\mathcal{O}(r^{c_{i_0}}));\qquad(4.25)$$

- if $S(i_0-1)<1$ and $S(i_0)=1$ then

$$\mathbb{E}C_T(r)=\left(\frac{2}{T}\left(\frac{2}{\pi}\right)^{i_0}\left(\frac{1}{\alpha_{i_0}}-\frac{1}{\alpha_{i_0+1}}\right)\prod_{k=1}^{i_0}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\right)r^{W(i_0)}\left(\log\frac{1}{r}+\mathcal{O}(1)\right).\qquad(4.26)$$

We finish by the case $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$. We deduce from the previous study that the expansion of $\mathbb{E}C_T(r)$ is obtained from its decomposition around $r^{1/\alpha_{i_0}}$. Thus, denote $j_0 = \min\{j \in \{1, \cdots, i_0\}, \alpha_j = \alpha_{i_0}\}$, $k_0 = \max\{k \in \{i_0, \cdots, d\}, \alpha_k = \alpha_{i_0}\}$ and let $\alpha_{j_0-1} < \beta_{j_0} < \alpha_{j_0} = \cdots = \alpha_{k_0} < \beta_{k_0+1} < \alpha_{k_0+1}$. From the previous computations, we deduce that:

$$\mathbb{E}C_T(r) = \frac{2}{T}\left(\frac{2}{\pi}\right)^{k_0}\left(\prod_{k=1}^{j_0-1}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\right) r^{W(j_0)}\left(1+\mathcal{O}(r^{c_{jO}})\right) \times ..$$

$$.. \times \int_{r^{\beta_{j_0}^{-1}-\alpha_{i_0}^{-1}}}^{r^{\beta_{k_0+1}^{-1}-\alpha_{i_0}^{-1}}} \frac{1}{v^{S(j_0-1)}}\left(\prod_{j=j_0}^{k_0}\int_0^\infty \frac{\sin\lambda}{\lambda}e^{-(a_j\lambda v^{\alpha_j})^{p_i}}d\lambda\right)dv$$

$$\mathbb{E}C_T(r) = \frac{2}{T}\left(\frac{2}{\pi}\right)^{k_0}\left(\int_0^\infty \frac{1}{v^{S(j_0-1)}}\left(\prod_{j=j_0}^{k_0}\int_0^\infty \frac{\sin\lambda}{\lambda}e^{-(a_j\lambda v^{\alpha_j})^{p_i}}d\lambda\right)dv\right) \times ..$$

$$.. \times \left(\prod_{k=1}^{j_0-1}\frac{1}{a_kp_k}\Gamma(\frac{1}{p_k})\right) r^{W(j_0)}\left(1+\mathcal{O}(r^{c_{jO}})\right),$$

which achieves the proof of the Lemma. ■

Lemma 4.5 *Under hypothesis of Proposition 3.2, there exist $l > 0$ and $m > 0$ depending only on $(\alpha_i)_i$, $(a_i)_i$, $(p_i)_i$ and T , such that for small enough r,*

$$\mathbb{E}\left(C_T(r) - \mathbb{E}C_T(r)\right)^2 \leq mr^{2l}\left(\mathbb{E}C_T(r)\right)^2. \tag{4.25}$$

Proof: By the same computations than in the proof of the previous lemma, and with $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}_+^d$ and $\lambda' = (\lambda'_1, \cdots, \lambda'_d) \in \mathbb{R}_+^d$, we have:

$$\mathbb{E}C_T^2(r) = \frac{1}{T^4}\int_{[0,T]^4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 2^{2d}\prod_{j=1}^{d}\frac{\sin(r\lambda_j)}{\lambda_j}\frac{\sin(r\mu_j)}{\mu_j} \times \cdots$$

$$\cdots \times \exp\left(-\parallel \lambda_j(X_t^{(j)} - X_s^{(j)}) + \lambda'_j(X_{t'}^{(j)} - X_{s'}^{(j)}) \parallel_{p_j}^{p_j}\right) d\lambda\, d\lambda'\, ds'dt'ds\, dt.$$

We follow now the same lines of the proof of Lemma 4.2. Let $q = (4\alpha_d)^{-1}$. From the Local Decorrelation assumption **(I5)**, denote

$$\Delta_r = \left\{(s,t,s',t') \in [0,T]^4 \text{ such that } |t-s| \leq r^{2q}\ ,\ |t'-s'| \leq r^{2q} \text{ and } |s-s'| \geq r^{2(1-\gamma)q}\right\}.$$

Thus, there is $r_0 > 0$, $\kappa > 0$ such that for each $j \in \{1, \cdots, d\}$, and all $(\lambda_j, \lambda'_j) \in \mathbb{R}_+^2$ such that $|\lambda_j| \leq r^{-q}$ and $|\lambda'_j| \leq r^{-q}$ and $(s,t,s',t') \in \Delta_r$, we have for $0 < r \leq r_0$:

$$\parallel \lambda_j(X_t^{(j)} - X_s^{(j)}) + \lambda'_j(X_{t'}^{(j)} - X_{s'}^{(j)}) \parallel_{p_j}^{p_j} \geq$$

$$\left(\parallel \lambda_j(X_t^{(j)} - X_s^{(j)}) \parallel_{p_j}^{p_j} + \parallel \lambda'_j(X_{t'}^{(j)} - X_{s'}^{(j)}) \parallel_{p_j}^{p_j}\right)\left(1 - \kappa\left(\frac{1}{r^{\delta q}}\right)\right).$$

More, from LND assumption **(I3)**, there exists $\chi_j > 0$ such that for all $(\lambda_j, \lambda'_j) \in \mathbb{R}_+^2$ and all $(s,t,s',t') \in [0,T]^4 \setminus \Delta_r$:

$$\| \lambda_j(X_t^{(j)} - X_s^{(j)}) + \lambda'_j(X_{t'}^{(j)} - X_{s'}^{(j)}) \|_{p_j}^{p_j} \geq \chi_j \left(\| \lambda_j(X_t^{(j)} - X_s^{(j)}) \|_{p_j}^{p_j} + \| \lambda'_j(X_{t'}^{(j)} - X_{s'}^{(j)}) \|_{p_j}^{p_j} \right).$$

Now, we decompose the previous expression of $\mathbb{E}C_T^2(r)$.
Firstly, denote the decomposition $\displaystyle\int_{[0,T]^4} = \int_{\Delta_r} + \int_{[0,T]^4 \setminus \Delta_r} = I_1 + I_2$.
Secondly, observe that for $0 < r \leq r_0$:

$$I_1 = \int_{\Delta_r} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} = \left(1 + \mathcal{O}\left(e^{-r^k}\right)\right) \int_{\Delta_r} \int_{[0,r^{-q}]^d} \int_{[0,r^{-q}]^d},$$

with $k > 0$, because for $0 \leq u \leq r^{2q}$,

$$\left| \int_{r^q}^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[- \left(\frac{a_i \lambda u^{\alpha_i}}{r} \right)_i^p \right] d\lambda \right| \leq \int_{r^q}^{\infty} \exp\left[- \left(a_i \lambda r^{-1/2} \right)_i^p \right] d\lambda.$$

Thus, we obtain for $0 < r \leq r_0$:

$$\begin{aligned} I_1 &= \left(1 + \mathcal{O}\left(e^{-kr}\right)\right)^{2d} \left(1 + \mathcal{O}\left(r^{\delta q}\right)\right) \frac{4}{T^2} \left(4^d \int_0^{r^2 q} du \prod_{j=1}^d \int_0^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[-\lambda^{p_j} \left(\frac{\sigma_j(u)}{r} \right)^{p_j} \right] d\lambda \right)^2 \\ &= \left(1 + \mathcal{O}\left(r^{\delta q}\right)\right) \left(\mathbb{E}C_T(r)\right)^2. \end{aligned}$$

Thirdly, the LND assumption **(I3)** implies:

$$\begin{aligned} I_2 = \left(1 + \mathcal{O}\left(e^{-kr}\right)\right)^d \frac{8.4^d}{T^2} &\left(\int_{r^2 q}^T du \prod_{j=1}^d \int_0^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[-\chi_j \lambda^{p_j} \left(\frac{\sigma_j(u)}{r} \right)^{p_j} \right] d\lambda \right) \times .. \\ &.. \times \left(\int_0^T du \prod_{j=1}^d \int_0^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[-\chi_j \lambda^{p_j} \left(\frac{\sigma_j(u)}{r} \right)^{p_j} \right] d\lambda \right). \end{aligned}$$

From computations of previous lemma and with $S(d) > 1$, we deduce that it exists $k' > 0$ such that for $0 < r \leq r_0$:

$$\begin{aligned} \int_{r^2 q}^T du \prod_{j=1}^d \int_0^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[-\chi_j \lambda^{p_j} \left(\frac{\sigma_j(u)}{r} \right)^{p_j} \right] d\lambda &= \mathcal{O}\left(r^{k'}\right) \mathbb{E}C_T(r) \quad \text{and} \\ \int_0^T du \prod_{j=1}^d \int_0^{\infty} \frac{\sin \lambda}{\lambda} \exp\left[-\chi_j \lambda^{p_j} \left(\frac{\sigma_j(u)}{r} \right)^{p_j} \right] d\lambda &= \mathcal{O}\left(\mathbb{E}C_T(r)\right). \end{aligned}$$

Thus,

$$I_2 = \mathcal{O}\left(r^{k'}\right) \left(\mathbb{E}C_T(r)\right)^2.$$

As a conclusion, and it achieves the proof, one deduces that for $S(d) > 1$

$$\mathbb{E}C_T^2(r) = \left(1 + \mathcal{O}\left(r^{2l}\right)\right)(\mathbb{E}C_T(r))^2$$

with $l > 0$. ■

Proof of Theorem 3.2: • First, if $S(d) > 1$ and following the same method than in the proof of Lemma 4.3: consider the sequence $v_n = \dfrac{C_T\left(\frac{1}{n^l}\right)}{\mathbb{E}C_T\left(\frac{1}{n^l}\right)}$, Borel-Cantelli Lemma provides $v_n \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 1$, and because $C_T(r)$ is an increasing function, the following equivalence is obtained:

$$C_T(r) \underset{r\to 0}{\overset{a.s.}{\sim}} \mathbb{E}C_T(r).$$

• Secondly, if $S(d) = 1$, we consider the same method than in the end of the proof of Proposition 3.1. ■

Proof of Theorem 3.1: Two cases have to be distinguished:

- case 1: if $\sum_{k=1}^{d} \alpha_k < 1$ we apply Property 2.4. In fact, Nolan in [15] shows that an index-α p-stable process verifying assumptions **(I1)**, **(I2')**, **(I3)** and **(I4')** has a jointly continuous local time on $[0, T]$ because it exists $\gamma > 0$ and $\varepsilon > 0$ such that

$$\int_{-\varepsilon}^{+\varepsilon}\int_{-\varepsilon}^{+\varepsilon}\prod_{j=1}^{d} \| X_t^{(j)} - X_s^{(j)} \|_{p_j}^{-(1+2\gamma)} \, dsdt,$$

since $\sum_{k=1}^{d} \alpha_k < 1$. More, this local time satisfies a Hölder condition in x of any order less than γ. As a consequence, *mutatis mutandis* in the proof of Property 2.4, ν exists and $\nu = d$ a.s.

- case 2: if $\sum_{k=1}^{d} \alpha_k \geq 1$, from assumption **(I2')**, for all $\varepsilon > 0$ there exists $t_0 > 0$ such that for each $i = 1, .., d$, for all $(s, t) \in [0, T]^2$ with $|t - s| \leq t_0$, we have:

$$a_i |t - s|^{\alpha_i + \varepsilon} \leq \| X_t^{(i)} - X_0^{(i)} \|_{p_i} \leq a_i |t - s|^{\alpha_i - \varepsilon}. \tag{4.26}$$

Assume $\alpha_1 \leq \cdots \leq \alpha_d$) and let $\varepsilon > 0$ be such that

$$\sum_{i=1}^{i_0} (\alpha_i + \varepsilon) > 1 \quad \text{and} \quad \sum_{i=1}^{i_0 - 1} (\alpha_i + \varepsilon) < 1. \tag{4.27}$$

For $j = 1, \cdots, d$, let $Y_t^j = (X_t^{(1)}, \cdots, X_t^{(j)})'$. It is obvious that for all $r \geq 0$ and $(s, t) \in [0, T]^2$,

$$\mathbb{I}_{(\|X_t - X_s\|_{\max} \leq r)} \leq \mathbb{I}_{(\|Y_t^j - Y_s^j\|_{\max} \leq r)},$$

As a consequence, for $n \in \mathbb{N}^*$,

$$\begin{aligned}
\mathbb{E}\left(C_T(r)\right)^n &= \frac{1}{T^{2n}}\int_{[0,T]^{2n}} \mathbb{E}\left(\prod_{i=1}^{n} \mathbb{I}_{(\|X_{t_i}-X_{s_i}\|_{\max}\le r)}\right) dsdt \\
&\le \frac{1}{T^{2n}}\sum_{j_1=1}^{i_0+1}\cdots\sum_{j_n=1}^{i_0+1}\int_{\bigotimes_{k=1}^n |t_k-s_k|\in A_{j_k}(r)} \mathbb{E}\left(\prod_{k=1}^{n}\mathbb{I}_{(\|Y_{t_k}^{j_k}-Y_{s_k}^{j_k}\|_{\max}\le r)}\right) dsdt \\
&\le \frac{1}{T^{2n}}\sum_{j_1=1}^{i_0+1}\cdots\sum_{j_1=0}^{i_0+1}\int_{\bigotimes_{k=1}^n |t_k-s_k|\in A_{j_k}(r)}\int_{\mathbb{R}^{j_1}}\cdots\int_{\mathbb{R}^{j_n}}\left(\prod_{k=1}^{n}\frac{\sin(r\lambda_k^{(j_k)})}{\lambda_k^{(j_k)}}\right)\times .. \\
&\qquad .. \times \mathbb{E}\left[\exp\left(i\sum_{k=1}^{n}\sum_{m=1}^{j_k}\lambda_k^{(m)}(X_{t_k}^{(m)}-X_{s_k}^{(m)})\right)\right] d\lambda dsdt,
\end{aligned}$$

with $A_i(r)=[r^{(\alpha_{i-1}+\varepsilon)^{-1}}, r^{(\alpha_i+\varepsilon)^{-1}}]$ for $i=1,\cdots,i_0$ and $A_{i_0+1}(r)=[r^{(\alpha_{i_0}+\varepsilon)^{-1}},T]$. But from assumptions **(I3)** and **(I4')**, it exists $c=c(n)>0$ such that for each $(j_1,\cdots,j_n)\in\{1,\cdots,i_0+1\}$ and all $\lambda_k^{(m)}\in[0,\infty)$,

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{n}\sum_{m=1}^{j_k}\lambda_k^{(m)}(X_{t_k}^{(m)}-X_{s_k}^{(m)})\right)\right] \le \prod_{k=1}^{n}\prod_{m=1}^{j_k}\exp\left(-c\parallel \lambda_k^{(m)}(X_{t_k}^{(m)}-X_{s_k}^{(m)})\parallel_{p_m}^{p_m}\right),$$

and thus from (4.26), for r small enough,

$$\begin{aligned}
\mathbb{E}\left(C_T(r)\right)^n &\le \left(\frac{2}{T}\right)^n\sum_{j_1=1}^{i_0+1}\cdots\sum_{j_n=1}^{i_0+1}\int_{\bigotimes_{k=1}^n u_k\in A_{j_k}(r)}\int_{\mathbb{R}^{j_1}}\cdots\int_{\mathbb{R}^{j_n}} r^{j_1+\cdots+j_n}\times .. \\
&\qquad .. \times\prod_{k=1}^{n}\prod_{m=1}^{j_k}\exp\left(-c(a_m\lambda_k^{(m)}u_k^{\alpha_m+\varepsilon})^{p_m}\right)d\lambda du.
\end{aligned}$$

From previous computations, if we denote $W_\varepsilon(j)=\dfrac{1+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j+\varepsilon}$,

$$\mathbb{E}\left(C_T(r)\right)^n \le K\left(\log\left(\frac{1}{r}\right)\right)^n\left(\sum_{j_1=1}^{i_0+1}\cdots\sum_{j_n=1}^{i_0+1} r^{W_\varepsilon(j_1)+\cdots+W_\varepsilon(j_n)}\right)$$

with a constant K only depending on $T, n, (a_i), (\alpha_i)$ and (p_i). As a consequence, for all $\eta>0$, it is always possible to chose ε and n such that for r small enough,

$$\mathbb{E}\left(C_T(r)\right)^n \le K'\, r^2\, r^{n(W(i_0)-\eta)},$$

with a constant K' only depending on $T, n, (a_i), (\alpha_i)$ and (p_i). Thus, from Tchebychev Inequality, for r small enough,

$$\mathbb{P}\left(\frac{C_T(r)}{r^{W(i_0)-\eta}}\ge 1\right)\le K'\, r^2.$$

Then, from Borel-Cantelli Lemma and the fact that $C_T(r)$ is an increasing function of r, we deduce that for all $\eta > 0$,

$$\limsup_{r\to 0} \frac{C_T(r)}{r^{W(i_0)-\eta}} \geq 1 \quad a.s.$$

More, we know from Property 2.2 that:

$$\liminf_{r\to 0} \frac{C_T(r)}{r^{W(i_0)+\eta}} \leq 1 \quad a.s.$$

and it finishes the proof. ∎

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