Invariance principle for new weakly dependent stationary models under sharp moment assumptions

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Abstract

This paper is aimed to sharpen a weak invariance principle for stationary sequences in Doukhan & Louhichi (1999). Our assumption is both beyond mixing and the causal θ-weak dependence in Dedecker and Doukhan (2003); those authors obtained a sharp result which improve on an optimal one in Doukhan *et alii* (1995) under strong mixing. We prove this result and we also precise convergence rates under existence of moments with order > 2 while Doukhan & Louhichi (1999) assume a moment of order > 4. Analogously to those authors, we use a non-causal condition to deal with some general classes of stationary and weakly dependent sequences. Beyond the previously used η- and κ-weak dependence conditions, we introduce a mixed condition λ adapted to consider Bernoulli shifts with dependent innovations, which look quite new.

1 Introduction and main results

Working with times series provide a huge amount of applications. Several ways of modeling the weak dependence have already been worked out. One of the most popular is the notion of mixing, see Doukhan (1994) for bibliography; this allows a very nice asymptotic theory (see Rio, 2000). However, using mixing presents lots of restrictions. For example, Andrews (1984) exhibits the simple counter-example of an auto-regressive process which does not satisfy a mixing condition. Doukhan and Louhichi (1999) introduced several new weak dependence conditions to solve those problems. The present work aims to provide a sharper Donsker theorem under such conditions. Among the notions of weak dependence, we consider separately causal and non causal conditions.

- On the one hand, causal Donsker's theorem is discussed in Dedecker & Doukhan (2003). Using arguments originated from martingale theory, those authors obtain sharp results exhibiting moment assumptions with order $2 + \zeta$ for arbitrarily small $\zeta > 0$.
- On the other hand, the case of non causal sequences yields other interesting models such as the two sided linear sequence

$$X_t = \sum_{-\infty}^{\infty} a_i \, \xi_{t-i} \tag{1}$$

where summations run from $-\infty$ to $+\infty$, and the innovations $(\xi_t)_{t\in\mathbb{Z}}$ form an iid sequence. In this case, no martingale theory tool seems to be available, and Doukhan and Louhichi (1999) obtain Donsker type invariance principle under moment conditions with order > 4. The result in [10] is clearly not optimal and the reason of such assumptions is that the method of proof was based on the Bernstein blocks technique and on a combinatorial moment inequality of order 4 which excludes lower order moment assumptions.

In the present paper, we propose an alternative to such results for which this problem is bypassed. We derive a moment inequality of order $2+\delta$, where $0 < \delta < \zeta$, for sums by using interpolation and truncation arguments. Our moment inequality writes $\|X_1 + \cdots + X_n\|_{2+\delta} \le c\sqrt{n}$. This expression allows to derive both the tightness in the Donsker invariance principle (see Billingsley, 1968) and a version of the central limit theorem under weak dependence conditions and lower order moments assumptions. We also introduce a composite weak dependence condition aimed to include new models such as the previous two sided linear models but where now the sequence of innovations, instead of being independent, may now be some weakly dependent process. The method used here, originated from Ibragimov (1975) was recently used by Bulinski and Sashkin (2005) for κ'-weakly dependent random fields (see the forthcoming definition).

Definition 1 *A vector valued process $(X_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^d$, endowed with some norm $\|\cdot\|$, is said to be (ϵ,ψ)-weakly dependent if there exists a sequence $\epsilon_r \downarrow 0$ (as $r \uparrow \infty$) and a function $\psi : \mathbb{N}^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ such that:*

$$|Cov(f(X_{s_1},\dots,X_{s_u}), g(X_{t_1},\dots,X_{t_v})| \le \psi(u,v,Lip\, f, Lip\, g)\epsilon_r,$$

for any $r \ge 0$ and any $(u+v)$-tuples such that $s_1 \le \cdots \le s_u \le s_u + r \le t_1 \le \cdots \le t_v$, where the real valued functions f, g are defined respectively on $\left(\mathbb{R}^d\right)^u$ and $\left(\mathbb{R}^d\right)^v$, and they satisfy $\|f\|_\infty, \|g\|_\infty \le 1$ and $Lip\, f + Lip\, g < \infty$ where we set

$$Lip\, f = \sup_{(x_1,\dots,x_u)\neq(y_1,\dots,y_u)} \frac{|f(x_1,\dots,x_u) - f(y_1,\dots,y_u)|}{\|x_1 - y_1\| + \cdots + \|x_u - y_u\|}$$

Specific interesting functions ψ are:

- *κ-weak dependence for which $\psi(u,v,a,b) = uvab$, in this case we simply denote ϵ_r as κ_r,*
- *κ' (causal) weak dependence for which $\psi(u,v,a,b) = vab$, in this case we simply denote ϵ_r as κ'_r; this is the causal counterpart of κ coefficients which we recall only for completeness,*
- *η-weak dependence, $\psi(u,v,a,b) = ua + vb$, in this case we write $\epsilon_r = \eta_r$ for short,*
- *θ-weak dependence is a causal dependence which refers to $\psi(u,v,a,b) = vb$, in this case we simply denote $\epsilon_r = \theta_r$ (see Dedecker & Doukhan, 2003); this is the causal counterpart of η coefficients which we recall only for completeness,*
- *λ-weak dependence $\psi(u,v,a,b) = uvab + ua + vb$, in this case we write $\epsilon_r = \lambda_r$.*

Remarks.

- Besides the fact that it includes η- and κ-weak dependence, λ-weak dependence will be proved to be convenient, for example, for Bernoulli shifts with associated innovations (see Theorem 3 below).
- If now the function f, g take their values in $\mathbb{C}$, the previous inequalities remain true by replacing ϵ_r by $\epsilon_r/2$. A useful case of such complex valued functions is $f(x_1, \dots, x_u) = \exp(it \cdot (x_1 + \cdots + x_u))$ for each $t \in \mathbb{R}^d$, $u \in \mathbb{N}^*$ and $x_1, \dots, x_u \in \mathbb{R}^d$ (see section 4.1). This indeed corresponds to the characteristic function adapted to derive convergence in distribution through Paul Lévy theorem.

In all the paper we shall consider a centered and stationary real valued sequence $(X_n)_{n\in\mathbb{Z}}$ such that

$$\mu = \mathbb{E}X_0^m < \infty, \qquad \text{for a real number } m = 2 + \zeta > 2. \tag{2}$$

We also set

$$\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k) = \sum_{k\in\mathbb{Z}} \mathbb{E}X_0 X_k,$$

W denotes the standard Brownian motion and $W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i$ for $t \in [0;1]$ and $n \geq 1$. We now present our main results, they provide a Donsker type weak invariance principle.

Theorem 1 (κ-dependence) *Assume that the κ-weakly dependent stationary process satisfies (2) and $\kappa_r = \mathcal{O}(r^{-\kappa})$ (as $r \uparrow \infty$) for $\kappa > 2 + \frac{1}{\zeta}$ then the previous expression $\sigma^2 \geq 0$ is well defined and, moreover:*

$$W_n(t) \to_{n\to\infty} \sigma W(t), \qquad \textit{in distribution in the Skohorod space } D([0,1]).$$

Remark. Under the more restrictive κ' condition, Bulinski & Shashkin (2005) obtain invariance principles with the sharper assumption $\kappa' > 1+1/\zeta$. The difference of 1 between both conditions is natural since it may be proved for κ'-weakly dependent sequences that $\kappa'_r \geq \sum_{s\geq r} \kappa_s$. This simple bound, checked from the definitions, explains the previous loss.

The following result relaxes the previous dependence assumptions to the coast of a faster decay for the dependence coefficients.

Theorem 2 (λ-dependence) *Assume that the λ-weakly dependent stationary process satisfies (2) and $\lambda_r = \mathcal{O}(r^{-\lambda})$ (as $r \uparrow \infty$) for $\lambda > 4 + \frac{2}{\zeta}$ then the previous expression $\sigma^2 \geq 0$ is well defined and, moreover:*

$$W_n(t) \to_{n\to\infty} \sigma W(t), \qquad \textit{in distribution in the Skohorod space } D([0,1]).$$

We do not achieve better results for η (or θ-) weak dependence cases than the one for $\lambda-$dependence. This last coefficient is very useful to study Bernoulli shifts $X_n = H(\xi_{n-j}, j \in \mathbb{Z})$ with weakly dependent innovation process $(\xi_i)_i$ (see section 2.4 for more details).

We restrict to such $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ satisfying:
for each $s \in \mathbb{Z}$, if $x, y \in \mathbb{R}^{\mathbb{Z}}$ satisfy $x_i = y_i$ for each index $i \neq s$

$$|H(x) - H(y)| \leq b_s(\max_{j\neq s} |x_j|^l \vee 1)|x_s - y_s|. \tag{3}$$

Theorem 3 *Let $(\xi_i)_i$ be a stationary λ-weakly dependent process (with dependence coefficients denoted $\lambda_{\xi,r}$) and $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ satisfies the condition (3) for some $m > 2$ such that $lm \le m' - 1$ with $\mathbb{E}|\xi_0|^{m'} < \infty$, and some sequence $b_i \ge 0$ such that $\sum_i |i| b_i < \infty$. Then $X_n = H(\xi_{n-i}, i \in \mathbb{Z})$ satisfies the weak invariance principle in the following cases:*

- ***Geometric case:*** *$b_r = \mathcal{O}\left(e^{-rb}\right)$ and $\lambda_{\xi,r} = \mathcal{O}\left(e^{-rc}\right)$.*
- ***Mixed case:*** *$b_r = \mathcal{O}\left(e^{-rb}\right)$ and $\lambda_{\xi,r} = \mathcal{O}\left(r^{-c}\right)$ with $c > 4 + 2/(2-m)$.*
- ***Riemanian case:*** *If $b_r = \mathcal{O}\left(r^{-b}\right)$ for some $b > 2$ and $\lambda_{\xi,r} = \mathcal{O}\left(r^{-c}\right)$ with*

$$c > \frac{(10-4m)b(m'-1)}{(2-m)(b-2)(m'-1-l)}.$$

The constants $b > 0$ and $c > 0$ obtained are different for each case.

This theorem is useful to derive the weak invariance principle in many cases. Section 2.4 will provide such examples, but the forthcoming one is already of interest:

Example 1

Consider the two sided sequence $X_t = \sum_{-\infty}^{\infty} a_i \, \xi_{t-i}$ with LARCH(∞) innovations:

$$\xi_t = \tilde{\xi}_t \left(a' + \sum_{j=1}^{\infty} a'_j \xi_{t-j} \right),$$

where the process $(\tilde{\xi}_i)_i$ is iid. Under Riemanian decays ($a_r = \mathcal{O}(r^{-a})$ and $a'_r = \mathcal{O}(r^{-a'})$), we derive from Theorem 3 the condition to obtain the weak invariance principle on the process $(X_t)_t$ as:

$$a' > \frac{(10-4m)a(m'-1)}{(2-m)(a-2)(m'-1-l)} + 1.$$

Remarks.

- The technique of the proofs is based on Lindeberg method and we prove in fact that $|\mathbb{E}\left(f(S_n/\sqrt{n}) - f(\sigma N)\right)| \le C n^{-c^*}$ for constants $c^*, C > 0$ and (f is here the characteristic function) depending only on the parameters ζ and κ or λ respectively and where $c^* < \frac{1}{2}$ (see Proposition 2 section 4.2 for more details). When κ or λ tends to infinity, we have $c^* = \zeta/(4+\zeta)$. For $\zeta \ge 2$ and κ or λ tends to infinity, we notice that $c^* \to \frac{1}{3}$.
- Using a smoothing lemma also yields an analogue bound for the Levy distance in the real case ($d = 1$):

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\frac{1}{\sqrt{n}} S_n \le t \right) - \mathbb{P}\left(\sigma N \le t\right) \right| \le C n^{-c'}.$$

  A first and easy way to control c' is to set $c' = c^*/4$ but the corresponding rate is really a bad one. Petrov (1995) obtains the exponent $\frac{1}{2}$ in the iid case and Rio (2000) reaches the exponent $\frac{1}{3}$ for strongly mixing sequences. In proposition 3 section 4.2, we achieve $c' = c^*/3$. Analogous non optimal convergence rates are also proved for the case of weakly dependent random fields in [9].

The paper is organized as follows. In the forthcoming section 2, we precise examples of models satisfying our assumptions. Besides more standard examples, we shall focus on examples of λ-weakly dependent sequences. The other sections are devoted to the proofs. We first derive conditions ensuring the convergence of the series σ^2 and we then obtain a bound for the moment of order $(2+\delta)$-th of a sum (of an independent interest), in section 3. The proofs are collected in section 4. The standard Lindeberg method with Bernstein blocks is developed in § 4.1. Rates of convergence for the Donsker's theorems are obtained in § 4.2.

2 Examples

Examples are classified by increasing complexity. In each cases, we discuss both the existence of a moment of order $m > 2$ and the weakly dependence properties. We first recall situations of Gaussian and associated sequences which satisfy a κ type dependence condition. After this we recall the properties of η-dependence for the stable Markov chains or the Bernoulli shift sequences. Those conditions are essentially already considered in Doukhan & Louhichi (1999); however some examples are updated, using for instance a work by Doukhan, Teyssière and Winant (2005) who consider the example of $LARCH(\infty)$ vector models. A final subsection introduces new models of Bernoulli shifts with dependent innovations. Their properties of weak dependence are also described; to our opinion, such models already justify the introduction of $\lambda-$weak dependence.

2.1 Associated and Gaussian models

The κ-weak dependence condition is known to hold for associated or Gaussian sequences. Recall that a process is associated if $\text{Cov}(f(X^{(n)}), g(X^{(n)})) \geq 0$ for any coordinatewise non-decreasing function $f, g : \mathbb{R}^n \to \mathbb{R}$ such that the previous covariance make sense with $X^{(n)} = (X_1, \ldots, X_n)$. In both cases this condition will hold with

$$\kappa_r = \sup_{j \geq r} |\text{Cov}(X_0, X_j)|$$

Notice the absolute values are needed only in the second case since for associated processes these covariances are nonnegative if they are finite. Independent sequences as well are associated and Pitt (1982) proves that a Gaussian process with nonnegative covariances is also associated. Finally, we quote that non-decreasing functions of associated sequences remain associated. This is a standard way to construct associated models from iid sequences (see e.g. Louhichi, 2001).

2.2 Stable Markov processes

This section is devoted to the study of the properties of stationary sequences satisfying a recurrence equation

$$X_n = F(X_{n-1}, \ldots, X_{n-d}, \xi_n)$$

where the sequence (ξ_n) is iid. In this case $Y_n = (X_n, \ldots, X_{n-d+1})$ is a Markov chain such that $Y_n = M(Y_{n-1}, \xi_n)$ with

$$M(x_1, \ldots, x_d, \xi) = (F(x_1, \ldots, x_d, \xi), x_1, \ldots, x_{d-1}).$$

Duflo (1996) proves in Theorem 1.IV.24 the stationarity of $(X_n)_n$ in $\mathbb{L}^m$ for $m > 2$ as soon as $\|F(0,\xi)\|_m < \infty$ and there exist a norm $\|\cdot\|$ on $\mathbb{R}^d$ and a real $0 \le a < 1$ such that $\|F(x,\xi) - F(y,\xi)\|_m \le a\|x-y\|$. In this setting it is simple to derive that θ-dependence holds with $\theta_r = \mathcal{O}(a^r)$ (as $r \uparrow \infty$) for the following examples:

- *Functional AR models:* $X_t = r(X_{t-1}, \ldots, X_{t-d}) + \xi_t$ if $\mathbb{E}|\xi_0|^m < \infty$ and $|r(u_1,\ldots,u_d) - r(v_1,\ldots,v_d)|^m \le \sum_{i=1}^d a_i|u_i - v_i|$ for some $a_1,\ldots,a_d \ge 0$ with $\sum_{i=1}^d a_i < 1$.

- *ARCH-type processes.* Here $d = 1$. Let $M(u,z) = A(u) + B(u)z$ for suitable Lipschitz functions $A(u), B(u), u \in \mathbb{R}$. The corresponding iterative model satisfies the previous relation if
$$a = \mathrm{Lip}\,(A) + \|\xi_0\|_m \mathrm{Lip}\,(B) < 1.$$
Examples of such Markov processes are nonlinear AR(1) processes ($B \equiv 1$), stochastic volatility models ($A \equiv 0$); classical ARCH(1) models ($A(u) = \alpha u,\ , B(u) = \sqrt{\beta + \gamma u^2}$ with $\alpha, \beta, \gamma \ge 0$).

- *Branching type models.* Here $d = 1$, and $D \ge 2$, set $\xi_t = \left(\xi_t^{(1)}, \ldots, \xi_t^{(D)}\right)$. Let now $A_1(u), \ldots, A_D(u), u \in \mathbb{R}$ be Lipschitz functions, and let
$$M\left(u, \left(z^{(1)}, \ldots, z^{(D)}\right)\right) = \sum_{j=1}^D A_j(u) z^{(j)}, \quad (u, z^{(1)}, \ldots, z^{(D)}) \in \mathbb{R}^{D+1}.$$
For such kernels, we also require $a = \sum_{j=1}^D \mathrm{Lip}\,(A_j)\mathbb{E}|\xi_0^{(j)}|^m < 1$.

The next section is devoted to non necessarily Markov models.

2.3 Bernoulli shifts

Let $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ be a measurable function. If the sequence $(\xi_n)_{n\in\mathbb{Z}}$ is independent and identically distributed on the real line, a Bernoulli shift with innovation process $(\xi_n)_{n\in\mathbb{Z}}$ is defined as
$$X_n = H\left((\xi_{n-i})_{i\in\mathbb{Z}}\right), \qquad n \in \mathbb{Z}.$$
The most simple case of infinitely dependent Bernoulli shift is the infinite two sided moving average process (1). Bernoulli shifts are $\eta-$weakly dependent (see Doukhan & Louhichi (1999)) with $\eta_r \le 2\delta_{[r/2]}$ where ($\|\cdot\|$ is a norm on $\mathbb{R}^d$):
$$\mathbb{E}\left\|H\left(\xi_j, j \in \mathbb{Z}\right) - H\left(\xi_j \mathbb{1}_{|j|\le r}, j \in \mathbb{Z}\right)\right\| \le \delta_r. \tag{4}$$
In order to apply our weak invariance principle, we shall need an additional moment assumption with order $m > 2$ which is not a direct consequence of this definition. This condition is checked in the forthcoming examples.

2.3.1 Chaotic Volterra models

A Volterra process is a stationary process defined through a convergent Volterra expansion
$$X_t = v_0 + \sum_{k=1}^\infty V_{k;t}, \text{ where } V_{k;t} = \sum_{-\infty < i_1 < \cdots < i_k < \infty} a_{k;i_1,\ldots,i_k} \xi_{t-i_1} \cdots \xi_{t-k}$$

where v_0 denotes a constant and $(a_{k;i_1,\ldots,i_k})_{(i_1,\ldots,i_k)\in\mathbb{Z}^k}$ are real numbers for each $k \geq 1$. This expression converges in $\mathbb{L}^m$ for $m \geq 1$, provided that $\mathbb{E}|\xi_0|^m < \infty$ and the weights satisfy $\sum_{k=1}^{\infty}\sum_{i_1<\cdots<i_k}|a_{k;i_1,\ldots,i_k}| < \infty$.

More general Volterra processes are also defined by expansions (possibly with repetitions):

$$X_t = v_0 + \sum_{k=1}^{\infty} V_{k;t}, \text{ where } V_{k;t} = \sum_{(i_1,\ldots,i_k)\in\mathbb{Z}^k} a_{k;i_1,\ldots,i_k}\xi_{t-i_1}\cdots\xi_{t-k}$$

where v_0 denotes a constant and $(a_{k;i_1,\ldots,i_k})_{(i_1,\ldots,i_k)\in\mathbb{Z}^k}$ are real numbers for each $k \geq 1$. This more involved expression converges in $\mathbb{L}^m$ as soon as $\sum_{k=1}^{\infty}\mathbb{E}|\xi_0|^{mk}\sum_{i_1<\cdots<i_k}|a_{k;i_1,\ldots,i_k}|^m < \infty$. Those models are η-dependent since δ_s is now the tail of the previous series.

The forthcoming examples are natural models for which such expansions may be proved.

2.3.2 $LARCH(\infty)$ models

A vast literature is devoted to the study of conditionally heteroskedastic models. A simple equation in terms of a vector valued process allows a unified treatment of those models, see [13]. Let $(\xi_t)_{t\in\mathbb{Z}}$ be an iid sequence of random $d \times D$-matrices, $(a_j)_{j\in\mathbb{N}^*}$ be a sequence of $D \times d$ matrices, and a be a vector in $\mathbb{R}^D$. A vector valued $LARCH(\infty)$ model is a solution of the recurrence equation

$$X_t = \xi_t\left(a + \sum_{j=1}^{\infty} a_j X_{t-j}\right) \tag{5}$$

We provide below sufficient conditions for the following chaotic expansion

$$X_t = \xi_t\left(a + \sum_{k=1}^{\infty}\sum_{j_1,\ldots,j_k\geq 1} a_{j_1}\xi_{t-j_1}a_{j_2}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}a\right) \tag{6}$$

Such $LARCH(\infty)$ models include a large variety of models, as

- Standard $LARCH(\infty)$ models, correspond to the case of real valued X_t and a_j.
- Bilinear model $X_t = \zeta_t\left(\alpha + \sum_{j=1}^{\infty}\alpha_j X_{t-j}\right) + \beta + \sum_{j=1}^{\infty}\beta_j X_{t-j}$ where the variables are real valued and ζ_t is the innovation. For this, we set

  $$\xi_t = \begin{pmatrix} \zeta_t \\ 1 \end{pmatrix}, a = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \text{ and } a_j = \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$$

  Expansion (6) coincides with the chaotic expansion in [15].
- GARCH(p,q) models,

  $$\begin{cases} r_t = \sigma_t\varepsilon_t \\ \sigma_t^2 = \sum_{j=1}^{p}\beta_j\sigma_{t-j}^2 + \gamma + \sum_{j=1}^{q}\gamma_j r_{t-j}^2 \end{cases}$$

  where $\gamma > 0$, $\gamma_i \geq 0$, $\beta_i \geq 0$ (and the variables ε are centered at expectation); for this, the previous bilinear model is written with $\alpha_0 = \frac{\gamma_0}{1-\sum\beta_i}$ et $\sum\alpha_i z^i = \frac{\sum\gamma_i z^i}{1-\sum\beta_i z^i}$ (see [15]).

- ARCH(∞) processes are given by equations,

$$\begin{cases} r_t = \sigma_t \varepsilon_t \\ \sigma_t^2 = \beta_0 + \sum\limits_{j=1}^{\infty} \beta_j r_{t-j}^2 \end{cases}$$

  One sets $\xi_t = \begin{pmatrix} \varepsilon_t & 1 \end{pmatrix}$, $a = \begin{pmatrix} \kappa\beta_0 \\ \lambda_1\beta_0 \end{pmatrix}$, $a_j = \begin{pmatrix} \kappa\beta_j \\ \lambda_1\beta_j \end{pmatrix}$ with $\lambda_1 = \mathbb{E}(\varepsilon_0^2)$, $\kappa^2 =$ Var (ε_0^2).

Endow the sets of matrices with a norm $\|\cdot\|$ of algebra, such as any norm for linear applications. Assume that $\Lambda = \|\xi_0\|_m \sum_{j\geq 1} \|a_j\| < 1$ then one stationary of solution of eqn. (5) in $\mathbb{L}^m$ is given by (6), it is essentially the only one (see Doukhan, Teyssière and Winant, 2005). The solution (6) of eqn. (5) is $\theta-$weakly dependent with

$$\theta_r = \left(\mathbb{E}\|\xi_0\| \sum_{k=1}^{r-1} k\Lambda^{k-1} A\left(\frac{r}{k}\right) + \frac{\Lambda^r}{1-\Lambda} \right) \mathbb{E}\|\xi_0\| \|a\|,$$

where $A(x) = \sum_{j\geq x} \|a_j\|$. There exists some constant $K > 0$ and $b, C > 0$ such that

$$\theta_r \leq \begin{cases} K \frac{(\log(r))^{b\vee 1}}{r^b}, \text{ under Riemaniann decay } A(x) \leq Cx^{-b}, \\ K(q\vee\Lambda)^{\sqrt{r}}, \text{ under geometric decay } A(x) \leq Cq^x. \end{cases}$$

2.3.3 Non-causal $LARCH(\infty)$ model

Now a_j is defined for $j \neq 0$ and the corresponding recurrence equation involves a summation for such $j \neq 0$. Doukhan, Teyssière and Winant (2005) prove the same results of existence as for the previous causal case (only replace summation for $j > 0$ by summation for $j \neq 0$) and the dependence is now of the η type with

$$\eta_r = \left(\|\xi_0\|_\infty \sum_{0\leq 2k<r} k\Lambda^{k-1} A\left(\frac{r}{2k}\right) + \frac{\Lambda^{r/2}}{1-\Lambda} \right) \mathbb{E}\|\xi_0\| \|a\|$$

where now

$$A(x) = \sum_{|j|\geq x} \|a_j\|, \qquad \Lambda = \|\xi_0\|_\infty \sum_{j\geq 1} \|a_j\| < 1.$$

Notice that a very restrictive new assumption is that innovations are uniformly bounded. The forthcoming example relaxes the assumption of independence for the innovations.

2.4 Bernoulli shifts with dependent innovations

Let us note $(\xi_i)_i$ the weakly dependent innovation process. We restrict to the case $d = 1$ in this section. Let $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be a measurable function and $X_n = H(\xi_{n-i}, i \in \mathbb{Z})$. Such models are proved to exhibit either λ- or η-weak dependence properties. Because Bernoulli shifts of κ-weak dependent innovations are neither κ- nor η-weakly dependent, the κ case is here included in the λ one. We assume that $\mathbb{E}|\xi_0|^{m'} < +\infty$ and we set here $\|x\| = \sup_{i\in\mathbb{Z}} |x_i|$. In order to study weak dependence properties of X_n, we assume that H satisfy condition (3), this is a stronger assumption that the one used in the case of independent innovations (see eqn. (4)). The following lemma both proves the existence and bounds the weak dependence properties of such models:

Lemma 1 *Let $X_n = H(\xi_{n-i}, i \in \mathbb{Z})$ be a Bernoulli shift such that $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ satisfies the condition (3) with $lm+1 < m'$ for some $m > 2$, with $\mathbb{E}|\xi_0|^{m'} < \infty$ and some sequence $b_s \geq 0$ such that $\sum_s |s| b_s < \infty$. Then,*

- *the process $X_n = H(\xi_{n-i}, i \in \mathbb{Z})$ is well defined in $\mathbb{L}^m$: this is a strongly stationary process.*
- *if the innovation process $(\xi_i)_{i\in\mathbb{Z}}$ is λ-weakly dependent (the weak dependence coefficients are denoted $\lambda_{\xi,r}$), then X_n is λ-weakly dependent with*

$$\lambda_k = c \inf_{r \leq [k/2]} \left(\sum_{i \geq r} |i| b_i \right) \vee \left((2r+1)^2 \lambda_{\xi,k-2r}^{\frac{m'-1-l}{m'-1}} \right).$$

- *if the innovation process $(\xi_i)_{i\in\mathbb{Z}}$ is η-weakly dependent (the weak dependence coefficients are denoted $\eta_{\xi,r}$) then X_n is η-weakly dependent and there exists a constant $c > 0$ such that*

$$\eta_k = c \inf_{r \leq [k/2]} \left(\sum_{i \geq r} |i| b_i \right) \vee \left((2r+1)^{1+\frac{l}{m'-1}} \eta_{\xi,k-2r}^{\frac{m'-2}{m'-1}} \right).$$

The proof is given in section 4.3. Specifying the decay rates, this lemma leads to a useful proposition:

Proposition 1 *For standard decays of the previous sequences, this is easy to get the following explicit bounds. Here $\lambda > 0$ and $\eta > 0$ are constants which can differ in each case.*

- *If $b_i = \mathcal{O}\left(i^{-b}\right)$ for some $b > 2$ and $\lambda_{\xi,i} = \mathcal{O}\left(i^{-\lambda}\right)$, resp. $\eta_{\xi,i} = \mathcal{O}\left(i^{-\eta}\right)$ (as $i \uparrow \infty$) then from a simple calculation, we optimize both terms in order to prove that $\lambda_k = \mathcal{O}\left(k^{-\lambda\left(1-\frac{2}{b}\right)\frac{m'-1-l}{m'-1}}\right)$, resp. $\eta_k = \mathcal{O}\left(k^{-\eta\frac{(b-2)(m'-2)}{(b-1)(m'-1)-l}}\right)$. Note that in the case $m' = \infty$ this exponent may be arbitrarily close to λ for large values of $b > 0$. This exponent may thus take all possible values between 0 and λ.*
- *If $b_i = \mathcal{O}\left(e^{-ib}\right)$ for some $b > 0$ and $\lambda_{\xi,i} = \mathcal{O}\left(e^{-i\lambda}\right)$, resp. $\eta_{\xi,i} = \mathcal{O}\left(e^{-i\eta}\right)$ (as $i \uparrow \infty$) we have $\lambda_k = \mathcal{O}\left(k^2 e^{-\lambda k \frac{b(m'-1-l)}{b(m'-1)+2\eta(m'-1-l)}}\right)$, resp. $\eta_k = \mathcal{O}\left(k^{\frac{m'-1-l}{m'-1}} e^{-\eta k \frac{b(m'-2)}{b(m'-1)+2\eta(m'-2)}}\right)$. The geometric decays for both $(b_i)_i$ and coefficients of the innovations ensure the geometric decay of the weak dependence coefficients associated to such a Bernoulli shift.*
- *If we consider now a common situation where the Bernoulli shift as geometric decay, say $b_i = \mathcal{O}\left(e^{-ib}\right)$ and $\lambda_{\xi,i} = \mathcal{O}\left(i^{-\lambda}\right)$, resp. $\eta_{\xi,i} = \mathcal{O}\left(i^{-\eta}\right)$ (as $i \uparrow \infty$) we find $\lambda_k = \mathcal{O}\left((\log k)^2 k^{-\lambda\frac{m'-1-l}{m'-1}}\right)$, resp. $\eta_k = \mathcal{O}\left((\log k)^{1+\frac{l}{m'-1}} k^{-\eta\frac{m'-2}{m'-1}}\right)$. If $m' = \infty$ this thus means that we only loose at most a factor $\log^2 k$ with respect to the dependence coefficients of the input dependent series $(\xi_i)_i$.*

After the simplest example 1, we now precise more involved examples of Bernoulli shifts with dependent innovations:

Example 2 (Volterra models with dependent inputs)

Consider the function H *defined by:*

$$H(x)=\sum_{k=0}^{K}\sum_{j_1,\dots,j_k}a^{(k)}_{j_1,\dots,j_k}x_{j_1}\cdots x_{j_k},$$

then if x,y *are as in eqn. (3):*

$$H(x)-H(y)=\sum_{k=1}^{K}\sum_{u=1}^{k}\sum_{\substack{j_1,\dots,j_{u-1}\\ j_{u+1},\dots,j_k}}a^{(k)}_{j_1,\dots,j_{u-1},s,j_{u+1},\dots,j_k}x_{j_1}\cdots x_{j_{u-1}}(x_s-y_s)x_{j_{u+1}}\cdots x_{j_k}.$$

¿From the triangular inequality we thus derive that suitable constants may be written as $l=K-1$ *and*

$$b_s=\sum_{k=1}^{K}\sum^{(k,s)}|a^{(k)}_{j_1,\dots,j_k}|$$

where $\sum^{(k,s)}$ *stands for the sums over all indices in* $\mathbb{Z}^k$ *where one of the indices* $j_1,\dots,j_k$ *takes the value* s *and*

$$L\equiv\sum_{k=0}^{K}\sum_{j_1,\dots,j_k}\left|a^{(k)}_{j_1,\dots,j_k}\right|.$$

Example 3 (Uniform Lipschitz Bernoulli shifts)

Assume that condition (3) holds with $l=0$*, then the previous result still hold. An example of such a situation is the case of LARCH(*∞*) non-causal processes with bounded (*$m'=+\infty$*) and dependent stationary innovations.*

3 Moments inequalities

Our proof for central limit theorems is based on a truncation method. For a truncation level $T\geq 1$ we shall denote $\overline{X}_k=f_T(X_k)-\mathbb{E}f_T(X_k)$ with $f_T(X)=X\vee(-T)\wedge T$. Let us simply remark that $\overline{X}_k$ has moments of any orders because it is bounded. Furthermore, for any $a\leq m$, we control the difference $\mathbb{E}|f_T(X_0)-X_0|^a$ by using Markov inequality:

$$\mathbb{E}|f_T(X_0)-X_0|^a\leq\mathbb{E}|X_0|^a\mathbb{1}_{\{|X_0|\geq T\}}\leq\mu T^{a-m},$$

thus Jensen inequality yields

$$\|\overline{X}_0-X_0\|_a\leq 2\mu^{\frac{1}{a}}T^{1-\frac{m}{a}}. \tag{7}$$

With this truncation, we are now in position to control both the limiting variance and the higher order moments.

3.1 Variances

Lemma 2 (Variances) *If one of the following conditions holds then the series σ^2 is convergent*

$$\sum_{k=0}^{\infty} \kappa_k < \infty \qquad (8)$$

$$\sum_{k=0}^{\infty} \lambda_k^{\frac{m-2}{m-1}} < \infty \qquad (9)$$

Proof. Using the fact that $\overline{X}_0 = g_T(X_0)$ is a function of X_0 with $\operatorname{Lip} g_T = 1$, $\|g_T\|_\infty \le 2T$ we derive

$$|\mathrm{Cov}(\overline{X}_0, \overline{X}_k)| \le \kappa_k \text{ or } 4T\lambda_k, \text{ respectively} \qquad (10)$$

In the κ dependence case, truncation can thus be omitted and

$$|\mathrm{Cov}(X_0, X_k)| \le \kappa_k \qquad (11)$$

we thus only consider λ dependence below; develop:

$$\mathrm{Cov}(X_0, X_k) = \mathrm{Cov}(\overline{X}_0, \overline{X}_k) + \mathrm{Cov}(X_0 - \overline{X}_0, X_k) + \mathrm{Cov}(\overline{X}_0, X_k - \overline{X}_k)$$

and using a truncation T to be determined we use the two previous bounds (7) and (10) with Hölder inequality with the exponents $\frac{1}{a} + \frac{4}{m} = 1$ to derive

$$\begin{aligned} |\mathrm{Cov}(\overline{X}_0, \overline{X}_k)| &\le 4T\lambda_k + 2\|X_0\|_m \|\overline{X}_0 - X_0\|_a \\ &\le 4T\lambda_k + 4\mu^{1/a+1/m} T^{1-m/a} \\ &\le 4(T\lambda_k + \mu T^{2-m}). \end{aligned}$$

Note that we used the relation $1 - m/a = 2 - m$. Thus using the truncation such that $T^{1+\zeta} = \frac{\mu}{\lambda_k}$ yields the bound

$$|\mathrm{Cov}(X_0, X_k)| \le 8\mu^{\frac{1}{\zeta+1}} \lambda_k^{\frac{\zeta}{\zeta+1}} = 8\mu^{\frac{1}{m-1}} \lambda_k^{\frac{m-2}{m-1}}. \qquad (12)$$

3.2 A $(2+\delta)$-order moment bound

Lemma 3 *Assume that the stationary and centered process $(X_i)_{i\in\mathbb{Z}}$ satisfies $\mathbb{E}|X_0|^{2+\zeta} < \infty$, and it is either κ-weakly dependent with $\kappa_r = \mathcal{O}(r^{-\kappa})$ or λ-weakly dependent with $\lambda_r = \mathcal{O}(r^{-\lambda})$. If either $\kappa > 2 + \frac{1}{\zeta}$, or $\lambda > 4 + \frac{2}{\zeta}$, then for all $\delta \in]0, A \wedge B \wedge 1[$ (where A and B are constants smaller than ζ and only depending of ζ and respectively κ or λ (see eqns. (16) and (17)), there exist $C > 0$ such that:*

$$\|S_n\|_\Delta \le C\sqrt{n}, \qquad \textit{where} \qquad \Delta = 2 + \delta.$$

Remarks.

- The constant C satisfies $C > \left(\dfrac{5}{2^{\delta/2} - 1}\right)^{1/\Delta} \sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)|$. Under the conditions of this lemma, the lemma 2 entails

$$c \equiv \sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)| < \infty.$$

- The result is sketched from Bulinski and Sashkin (2005); notice, however that their condition of dependence is of a causal nature while our is not which explains a loss with respect to the exponents λ and κ. In their $\kappa'-$weak dependence setting the best possible value of the exponent is 1 while this is 2 here; a remark after the theorem explains this difference.

Proof of lemma 3. Analogously to Bulinski and Sashkin (2005), who extend Ibragimov (1979)'s proof to the case of random fields, we proceed by recurrence on k for $n \le 2^k$ to prove the property:

$$\|1 + |S_n|\|_\Delta \le C\sqrt{n}. \tag{13}$$

We then assume (13) for all $n \le 2^{K-1}$. We note $N = 2^K$ and we want to bound $\|1+|S_N|\|_\Delta$. We divide the sum S_N in three blocks, the two first with the same size $n \le 2^{K-1}$ are denoted A and B, and the third V placed between the two first has the cardinal $q < n$. We then have $\|1+|S_N|\|_\Delta \le \|1+|A|+|B|\|_\Delta + \|V\|_\Delta$. The term $\|V\|_\Delta$ is directly bounded by $\|1+|V|\|_\Delta$ and the recurrence assumption, i.e. $C\sqrt{q}$. If $q = N^b$ with $b < 1$, this term is of a strictly smaller order than $\sqrt{N}$. For $\|1+|A|+|B|\|_\Delta$, we have:

$$\begin{aligned}
\mathbb{E}(1+|A|+|B|)^\Delta &\le \mathbb{E}(1+|A|+|B|)^2(1+|A|+|B|)^\delta,\\
&\le \mathbb{E}(1+2|A|+2|B|+(|A|+|B|)^2(1+|A|+|B|)^\delta.
\end{aligned}$$

An expansion yields the terms:

- $\mathbb{E}(1+|A|+|B|)^\delta \le 1 + |A|_2^\delta + |B|_2^\delta \le 1 + 2c^\delta(\sqrt{n})^\delta$,
- $\mathbb{E}|A|(1+|A|+|B|)^\delta \le \mathbb{E}|A|((1+|B|)^\delta + |A|^\delta) \le \mathbb{E}|A|(1+|B|)^\delta + \mathbb{E}|A|^{1+\delta}$. The term $\mathbb{E}|A|^{1+\delta}$ is bounded by $\|A\|_2^{1+\delta}$ and then $c^{1+\delta}(\sqrt{n})^{1+\delta}$. The term $\mathbb{E}|A|(1+|B|)^\delta$ is bounded using Hölder $\|A\|_{1+\delta/2}\|1+|B|\|_\Delta^\delta$ and then is at least of order $cC^\delta(\sqrt{n})^{1+\delta}$.
- We have the analogous phenomenon if B is considered in place of A.
- $\mathbb{E}(|A|+|B|)^2(1+|A|+|B|)^\delta$. For this term, we use the elementary inequality from Bulinski & Sashkin:

  $$\mathbb{E}(|A|+|B|)^2(1+|A|+|B|)^\delta \le \mathbb{E}|A|^\Delta + \mathbb{E}|B|^\Delta + 5(\mathbb{E}A^2(1+|B|)^\delta + \mathbb{E}B^2(1+|A|)^\delta).$$

  Using (13), the term $\mathbb{E}|A|^\Delta$ is bounded by $C^\Delta(\sqrt{n})^\Delta$. The second term is analogous. The third is treated with a particular care below.

We now want to control $\mathbb{E}A^2(1+|B|)^\delta$ and the analogous with B. For this, we use the weak dependence. We thus have to truncate the variables. Denote by $\overline{X}$ the variable $X \vee T \wedge T$ for a real T that will determined later. We also set $\overline{A}$ and $\overline{B}$ for the analogue sums of the truncated variables $\overline{X}_i$. Remarking that $|B| - |\overline{B}| \ge 0$, we have:

$$\mathbb{E}|A|^2(1+|B|)^\delta \le \mathbb{E}A^2(|B|-|\overline{B}|)^\delta + \mathbb{E}(A^2 - \overline{A}^2)(1+|\overline{B}|)^\delta + \mathbb{E}\overline{A}^2(1+|\overline{B}|)^\delta.$$

We first control $\mathbb{E}A^2(|B|-|\overline{B}|)^\delta$. Set $m = 2+\zeta$, then using Hölder inequality with $2/m + 1/m' = 1$ yields:

$$\mathbb{E}A^2(|B|-|\overline{B}|)^\delta \le \|A\|_m^2\|(|B| - \|\overline{B}|)^\delta\|_{m'}$$

$\|A\|_\Delta$ is bounded using (13) and we remark that:

$$(|B| - |\overline{B}|)^{\delta m'} \leq (|B| - |B|\mathbb{1}_{\{\forall i, |X_i| \leq T\}})^{\delta m'} \leq |B|^{\delta m'} \mathbb{1}_{\{\exists i, |X_i| > T\}} \leq |B|^{\delta m'} \mathbb{1}_{|B|>T}.$$

We then bound $\mathbb{1}_{|B|>T} \leq (|B|/T)^\alpha$ with $\alpha = m - \delta m'$. Then

$$\mathbb{E}||B| - |\overline{B}||^{\delta m'} \leq E|B|^m T^{\delta m' - m}.$$

Then, by convexity and stationarity, we have $\mathbb{E}|B|^m \leq n^m \mathbb{E}|X_0|^m$. Then:

$$\mathbb{E}A^2(|B| - |\overline{B}|)^\delta \preceq n^{2+m/m'} T^{\delta - m/m'}.$$

Finally, remarking that $m/m' = m - 2$, we obtain:

$$\mathbb{E}A^2(|B| - |\overline{B}|)^\delta \preceq n^m T^{\Delta - m}.$$

We obtain the same bound for the second term:

$$\mathbb{E}(A^2 - \overline{A}^2)(1 + |\overline{B}|)^\delta \preceq n^m T^{\Delta - m}.$$

For the third term, we introduce a covariance term:

$$\mathbb{E}\overline{A}^2(1 + |\overline{B}|)^\delta \leq \mathrm{Cov}(\overline{A}^2, (1 + |\overline{B}|)^\delta) + \mathbb{E}\overline{A}^2 \mathbb{E}(1 + |\overline{B}|)^\delta.$$

The last term is bounded with $|A|_2^2 |B|_2^\delta \leq c^\Delta \sqrt{n}^\Delta$. The covariance is bounded above by using weak-dependence:

- in the κ-dependent case: $n^2 T \kappa_q$,
- in the λ-dependent case: $n^3 T^2 \lambda_q$.

We then choose either the truncation $T^{m-\delta-1} = n^{m-2}/\kappa_q$ or $T^{m-\delta} = n^{m-3}/\lambda_q$. Now the three terms of the decomposition have the same order:

$$\begin{aligned}
\mathbb{E}|A|^2(1 + |B|)^\delta &\preceq \left(n^{3m-2\Delta} \kappa_q^{m-\Delta}\right)^{1/(m-\delta-1)} && \text{under } \kappa\text{-dependence,} \\
\mathbb{E}|A|^2(1 + |B|)^\delta &\preceq \left(n^{5m-3\Delta} \lambda_q^{m-\Delta}\right)^{1/(m-\delta)} && \text{under } \lambda\text{-dependence.}
\end{aligned}$$

Set $q = N^b$, we note that $n \leq N/2$ and this term has order $N^{\frac{3m-2\Delta+b\kappa(\Delta-m)}{m-\delta-1}}$ under κ-weak dependence and the order $N^{\frac{5m-3\Delta+b\lambda(\Delta-m)}{m-\delta}}$ under λ-weak dependence. Those terms are thus negligible with respect to $N^{\Delta/2}$ if:

$$\kappa > \frac{3m - 2\Delta - \Delta/2(m - \delta - 1)}{b(m - \Delta)}, \qquad \text{under } \kappa\text{-dependence,} \tag{14}$$

$$\lambda > \frac{5m - 3\Delta - \Delta/2(m - \delta)}{b(m - \Delta)}, \qquad \text{under } \lambda\text{-dependence.} \tag{15}$$

Finally, using this assumption, $b < 1$ and $n \leq N/2$ we derive the bound for some suitable constants $a_1, a_2 > 0$:

$$\mathbb{E}(1 + |S_N|)^\Delta \leq \left(2^{-\delta/2} C^\Delta + 5 \cdot 2^{-\delta/2} c^\Delta + a_1 N^{-a_2}\right) \left(\sqrt{N}\right)^\Delta.$$

Using the relation between C and c, we conclude that (13) is also true at the step N if the constant C satisfies $2^{-\delta/2}C^\Delta + 5\cdot 2^{-\delta/2}c^\Delta + a_1 N^{-a_2} \le C^\Delta$. Choose $C > \left(\frac{5c^\Delta + a_1 2^{\delta/2}}{2^{\delta/2}-1}\right)^{1/\Delta}$ with $c = \sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)|$, then the previous relation holds.
Finally, we use eqns. (14) and (15) to derive a condition on δ. In the case of κ-weak dependence, we rewrite inequality (14) as:

$$0 > \delta^2 + \delta(2\kappa - 3 - \zeta) - \kappa\zeta + 2\zeta + 1.$$

It leads to the following condition on λ:

$$\delta < \frac{\sqrt{(2\kappa-3-\zeta)^2 + 4(\kappa\zeta - 2\zeta - 1)} + \zeta + 3 - 2\kappa}{2} = A. \tag{16}$$

We do the same in the case of the λ-weak dependence:

$$\delta < \frac{\sqrt{(2\lambda-6-\zeta)^2 + 4(\lambda\zeta - 4\zeta - 2)} + \zeta + 6 - 2\lambda}{2} = B. \tag{17}$$

Remark: those bounds are always smaller than ζ. □

4 Proof of the main results

In this section we first prove the central limit corresponding to the convergence $W_n(1) \to W(1)$ in the theorems 1 and 2, then we give rates for this central limit results. The weak invariance principle is obtained in a standard way from such central limit theorems and tightness which follows from lemma 1, by using the classical Kolmogorov Centsov tightness criterion, see Billingsly (1968).

In the last subsection, we prove the lemma 3 which states the properties of our (new) Bernoulli shifts with dependent inputs.

4.1 Proof of Theorems 1 & 2

Set $S = \frac{1}{\sqrt{n}} S_n$ and consider $p = p(n)$ and $q = q(n)$ such that

$$\lim_{n\to\infty} \frac{1}{q(n)} = \lim_{n\to\infty} \frac{q(n)}{p(n)} = \lim_{n\to\infty} \frac{p(n)}{n} = 0$$

and $k = k(n) = \left[\frac{n}{p(n)+q(n)}\right]$

$$Z = \frac{1}{\sqrt{n}}\left(U_1 + \cdots + U_k\right), \qquad \text{with } U_j = \sum_{i\in B_j} X_i$$

where $B_j =](p+q)(j-1), (p+q)(j-1)+p] \cap \mathbb{N}$ is a subset of p successive integers from $\{1,\ldots,n\}$ such that, for $j \neq j'$, B_j and $B_{j'}$ are at least distant of $q = q(n)$. We note B'_j the block between B_j and B_{j+1} and $V_j = \sum_{i\in B_j} X_i$. V_k is the last block of X_i between the end of B_k and n. Furthermore, set $\sigma_p^2 = \mathrm{Var}\,(U_1)/p$, we let

$$Y = \frac{V'_1 + \cdots + V'_k}{\sqrt{n}}, \qquad V'_j \sim \mathcal{N}(0, p\sigma_p^2)$$

where the Gaussian variables V_j are independent and independent of the sequence $(X_n)_{n\in\mathbb{Z}}$. We also consider a sequence $U_1^*,\dots,U_k^*$ of independent random variables with the same distribution as U_1 and we set $Z^*=\frac{1}{\sqrt{n}}(U_1^*+\cdots+U_k^*)$. We fix $t\in\mathbb{R}^d$ and we define $f:\mathbb{R}^d\to\mathbb{C}$ with $f(x)=\exp\{it\cdot x\}$. Then:

$$\mathbb{E}f(S)-f(\sigma N)=\mathbb{E}f(S)-f(Z)+\mathbb{E}f(Z)-f(Z^*)+f(Z^*)-f(Y)+\mathbb{E}f(Y)-f(\sigma N)$$

Lindeberg method will prove that this expression converges to 0 as $n\to\infty$. The first and the last term in this inequality are referred to as auxiliary terms in this Bernstein-Lindeberg method. They come from the replaced of the individual initial - non-Gaussian and Gaussian respectively - random variables. The second term is analogue to that obtained with decoupling and turns the proof of the central limit theorem to the independent case. The third term is referred to as the main term and following the proof under independence it will be bounded above by using a Taylor expansion. Because of the dependence structure, in the corresponding bounds, some additional covariance terms will appear.
The following subsections are thus organized to following this scheme: we first consider the auxiliary terms and the main terms are then decomposed by the usual Lindeberg method and the corresponding terms coming from the dependence or the usual remainder terms (standard for the independent case) are considered in separated subsections. A last subsection makes use of those individual calculations to derive the central limit theorem.

4.1.1 Auxiliary terms

Using Taylor expansions up to the second order, we bound the auxiliary terms:

$$\begin{aligned}|\mathbb{E}f(S)-f(Z)| &\le \frac{\|f''\|_\infty^2}{2}\mathbb{E}|S-Z|^2 \qquad \text{and,}\\ |\mathbb{E}f(Y)-f(\sigma N)| &\le \frac{\|f''\|_\infty^2}{2}\mathbb{E}|Y-\sigma N|^2\end{aligned}$$

Firstly:

$$\mathbb{E}|Z-S|^2\preceq\frac{\mathbb{E}\left(V_1+\ldots+V_k\right)^2}{n}$$

Note that the set under the sums of X_i in $V_1+\ldots+V_k$ has cardinality $\le(k+1)q+p$. Then we derive with (12) and (11) that under conditions (9) or (8) (respectively):

$$\mathbb{E}|Z-S|^2\preceq\frac{(k+1)q+p}{n}$$

We notice that Y follows the distribution of $\sqrt{\frac{kp}{n}}\sigma_pN$ and then working with Gaussian random variables:

$$\mathbb{E}|Y-\sigma N|^2 \;\le\; \left|\frac{kp}{n}-1\right|\sigma_p^2+\left|\sigma_p^2-\sigma^2\right|$$

Remarking that $kp/n-1\preceq q/p$ e need to bound

$$|\sigma_p^2-\sigma^2| \;\le\; \sum_{|i|<p}\frac{|i|}{p}|\mathbb{E}X_0X_i|+\sum_{|i|>p}|\mathbb{E}X_0X_i|$$

Set $a_i = |\mathbb{E}X_0X_i|$, under conditions (9) or (8) (respectively), the series $\sum_{i=0}^{\infty} a_i$ converge and $s_j = \sum_{i=j}^{\infty} a_i \to_{j\to\infty} 0$ then

$$|\sigma_p^2 - \sigma^2| \le 2\sum_{i=0}^{p-1} \frac{i}{p} \cdot a_i + 2s_p \le \frac{2}{p}\sum_{i=0}^{p-1} s_i + 2s_p,$$

Cesaro lemma entails that this term converges to 0. Hence $|\mathbb{E}f(S) - f(Z)| + |\mathbb{E}f(Y) - f(\sigma N)|$ tends to 0 as $n \uparrow \infty$.
To precise a rate of convergence, we assume that $a_i = \mathcal{O}(i^{-\alpha})$ with $\alpha > 1$; then w

$$|\sigma_p^2 - \sigma^2| \preceq p^{1-\alpha}.$$

The convergence rate is thus given by $\frac{q}{p} + \frac{p}{n} + p^{1-\alpha}$ if $\mathbb{E}X_0X_i = \mathcal{O}(i^{-\alpha})$. Remarking that $\mathbb{E}X_0X_i = \mathrm{Cov}(X_0, X_i)$, we then use equations (11) and (12) and we find $\alpha = \kappa$ or $\alpha = \lambda(m-2)/(m-1)$ depending of the weak-dependence setting.

With $p = n^a$, $q = n^b$, those bounds become:

$n^{b-a} + n^{a-1} + n^{a(1-\kappa)}$, in the κ-weak dependence setting,

$n^{b-a} + n^{a-1} + n^{a(1-\lambda(m-2)/(m-1))}$, under λ-weak dependence.

4.1.2 Main terms

It remains to control the second and the third terms of the sum. As usual in the Lindeberg technique:

$$|\mathbb{E}f(Z) - f(Z^*)| \le \textstyle\sum_{j=1}^{k} |\mathbb{E}\Delta_j|,$$
$$|\mathbb{E}f(Z^*) - f(Y)| \le \textstyle\sum_{j=1}^{k} |\mathbb{E}\Delta'_j|,$$
where $\Delta_j = f(W_j + x_j) - f(W_j + x_j^*), \quad j = 1, \ldots, k$ with
$$x_j = \tfrac{1}{\sqrt{n}}U_j, \quad x_j^* = \tfrac{1}{\sqrt{n}}U_j^*, \quad W_j = w_j + \textstyle\sum_{i>j} x_i^*, \quad w_j = \sum_{i<j} x_i$$
and $\Delta'_j = f(W'_j + x_j^*) - f(W'_j + x'_j), \quad j = 1, \ldots, k$ with
$$x'_j = \tfrac{1}{\sqrt{n}}V'_j, \quad W'_j = \textstyle\sum_{i<j} x_i^* + \sum_{i>j} x'_i.$$

Now we use the special form of f and the independence properties of the variables U_i^* and V'_i to write:

$$\begin{aligned}
\mathbb{E}\Delta_j &= \left(\mathbb{E}f(w_j)f(x_j) - \mathbb{E}f(w_j)\mathbb{E}f(x_j^*)\right)\mathbb{E}f\left(\sum_{i>j} x_i^*\right),\\
\mathbb{E}\Delta_j &= \left(\mathbb{E}f(x_j^*) - \mathbb{E}f(x'_j)\right)\mathbb{E}f\left(W'_j\right).
\end{aligned}$$

We then control the terms $\mathbb{E}f\left(\sum_{i>j} x_i^*\right)$ and $\mathbb{E}f\left(W'_j\right)$ by the fact that $\|f\|_\infty \le 1$ and we introduce a coupling to obtain:

$$\begin{aligned}
|\mathbb{E}\Delta_j| &\le \left|\mathrm{Cov}\left(f\left(\sum_{i<j} x_i\right), f(x_j)\right)\right|,\\
|\mathbb{E}\Delta_j| &= \left|\mathbb{E}f(x_j^*) - \mathbb{E}f(x'_j)\right|.
\end{aligned}$$

- For Δ_j, we use the weak-dependence definition to control this bound rewriting $|\mathrm{Cov}(F(X_m, m \in B_i, i < j), G(X_m, m \in B_j))|$, remarking that $\|F\|_\infty \le 1$ and the control of $\mathrm{Lip}\, F$ for $F(x_1, \dots, x_{kp}) = f\left(\frac{1}{\sqrt{n}} \sum_{i<j} x_i\right)$ (with possible repetitions in the sequence $(x_1, \dots, x_{kp})$):

$$\begin{aligned}\left| f\left(\frac{1}{\sqrt{n}} \sum_{i<j} x_i\right) - f\left(\frac{1}{\sqrt{n}} \sum_{i<j} y_i\right)\right| &\le \left|1 - \exp it \cdot \left(\frac{1}{\sqrt{n}} \sum_{i<j} (y_i - x_i)\right)\right| \\ &\le \frac{\|t\|_2}{\sqrt{n}} \sum_{i=1}^{kp} \|y_i - x_i\|_2.\end{aligned}$$

  For $G(x_1, \dots, x_p) = f\left(\sum_{i=1}^p x_i/\sqrt{n}\right)$, we have $\|G\|_\infty = 1$ and $\mathrm{Lip}\, G \preceq 1/\sqrt{n}$. We then distinguish the two cases, remarking the gap between the left and the right terms in the covariance is more than q:

  - In the κ dependence setting:

$$|\mathbb{E}\Delta_j| \preceq kp \cdot p \cdot \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} \cdot \kappa_q.$$

  - In the λ dependence setting:

$$|\mathbb{E}\Delta_j| \preceq \left(kp \cdot p \cdot \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} + kp \cdot \frac{1}{\sqrt{n}} + p \cdot \frac{1}{\sqrt{n}}\right) \cdot \lambda_q.$$

  Remarking the bounds do not depend of j, we then control the contribution of the second term as:

$$\begin{aligned}|\mathbb{E}f(Z) - f(Z^*)| &\preceq kp \cdot \kappa_q, \text{ under } \kappa, \\ &\preceq kp(1 + \sqrt{k/p}) \cdot \lambda_q, \qquad \text{under } \lambda.\end{aligned}$$

  Reminding that $p = n^a$, $q = n^b$, $\kappa_r = \mathcal{O}\left(r^{-\kappa}\right)$ or $\lambda_r = \mathcal{O}\left(r^{-\lambda}\right)$, the bounds become $n^{1-\kappa b}$ or $n^{1+(1/2-a)_+ - \lambda b}$ in respectively κ or λ context.

- For Δ'_j, writing Taylor formula up to order 2 or 3 respectively yields:

$$\begin{aligned}f(x_j^*) - f_j(x_j') &\le (x_j^* - x_j')\|f'\|_\infty + \frac{1}{2}(x_j^* - x_j')^2 \|f''\|_\infty + r_j \\ |r_j| &\le \frac{1}{2}\|f''\|_\infty (x_j^* - x_j')^2, \text{ or} \\ &\le \frac{1}{6}\|f'''\|_\infty |x_j^* - x_j'|^3,\end{aligned}$$

  We then have for an arbitrary $\delta \in [0, 1]$:

$$\begin{aligned}\mathbb{E}|r_j| &\preceq \mathbb{E}((|x_j^*|^2 + |x_j'|^2) \wedge (|x_j^*|^3 + |x_j'|^3)) \\ &\preceq \mathbb{E}(|x_j^*|^2 \wedge |x_j^*|^3) + \mathbb{E}(|x_j'|^2 \wedge |x_j'|^3) \\ &\preceq \mathbb{E}|x_j^*|^{2+\delta} + \mathbb{E}|x_j'|^{2+\delta}.\end{aligned}$$

Then, using the stationarity of the sequence X_n we obtain:

$$|\mathbb{E}\Delta_j'| \quad \preceq \quad n^{-1-\frac{\delta}{2}} \left(\mathbb{E}|S_p|^{2+\delta} \vee p^{1+\frac{\delta}{2}}\right).$$

We then use the result of lemma 3 to bound the moment $\mathbb{E}|S_p|^{2+\delta}$. If $\kappa > 2 + \frac{1}{\zeta}$, or $\lambda > 4 + \frac{2}{\zeta}$, where $\kappa_r = \mathcal{O}\left(r^{-\kappa}\right)$ or $\lambda_r = \mathcal{O}\left(r^{-\lambda}\right)$ then there exists $\delta \in]0, \zeta \wedge 1[$ and $C > 0$ such that:

$$\mathbb{E}|S_p|^{2+\delta} \leq Cp^{1+\delta/2}.$$

We then obtain:

$$|\mathbb{E}f(Z^*) - f(Y)| \preceq k(p/n)^{1+\delta/2}.$$

Reminding that $p = n^a$, the bound is of order $n^{(a-1)\delta/2}$ in both κ or λ-weak dependence setting.

4.2 Rates of convergence

We present two propositions that give rates of convergence in the weak invariance principle.

Proposition 2 *Assume that the weakly dependent stationary process $(X_n)_n$ satisfies (2) then the difference between the characteristic functions is bounded by ($C > 0$ is a constant):*

$$\left|\mathbb{E}\left(f(S_n/\sqrt{n}) - f(\sigma N)\right)\right| \leq Cn^{-c^*},$$

where c^* depends of the weakly dependent coefficients:

- *in the λ-dependence case, assume that $\lambda_r = \mathcal{O}(r^{-\lambda})$ for $\lambda > 4 + \frac{2}{\zeta}$, then $c^* = \dfrac{A}{2}\dfrac{2\lambda - 1}{(2+A)(\lambda+1)}$ where*

$$A = \frac{\sqrt{(2\lambda - 6 - \zeta)^2 + 4(\lambda\zeta - 4\zeta - 2)} + \zeta + 6 - 2\lambda}{2} \wedge 1,$$

- *in the κ-dependence case, if $\kappa_r = \mathcal{O}(r^{-\kappa})$ for $\kappa > 2 + \frac{1}{\zeta}$, then $c^* = \dfrac{(\kappa-1)B}{\kappa(2+B)}$ where*

$$B = \frac{\sqrt{(2\kappa - 3 - \zeta)^2 + 4(\kappa\zeta - 2\zeta - 1)} + \zeta + 3 - 2\kappa}{2} \wedge 1.$$

We restrict us in the case $d = 1$ and we use Theorem 5.1 of Petrov (1995) to obtain:

Proposition 3 (A rate in the Berry Essen bounds) *Assume that the real weakly dependent stationary process $(X_n)_n$ satisfies the same assumptions than in Proposition 2. We obtain:*

$$\sup_x |F_n(x) - \Phi(x)| = \mathcal{O}\left(n^{-c^*/3}\right).$$

where c^* is defined in Proposition 2.

Proof of proposition 2. In the previous section, we have expressed the rates of the different terms. We recall here them:

- In the λ-dependence case, we finally only have to consider the three largest rates: $(a-1)\delta/2$, $1+(1/2-a)_+-\lambda b$ and $b-a$. The previous optimal choice of a^* is smaller than $1/2$, then we have to consider the rate $3/2-a-\lambda b$ and not $1-\lambda b$. Then we find:

$$\begin{aligned} a^* &= \frac{(1+\lambda)\delta+3}{(2+\delta)(\lambda+1)} \in]0;1/2[\\ b^* &= a^* \frac{3}{2(\lambda+1)} \in]0;a^*[\end{aligned}$$

  Finally, we obtain the rate n^{-c^*}.

- In the κ-dependence case:
  - Auxiliary terms: $b-a$, $a-1$ and $a(1-\kappa)$,
  - Main terms: $1-\kappa b$ and $(a-1)\delta/2$.

  The idea is to choose carefully a^* and $b^*\in]0;1[$ such that the main rates are equal. Because $\delta<1$, $a>b$, we directly see that $(a-1)\delta/2>a-1$ and $1-\kappa b>a(1-\kappa)$, so that the only rate of the auxiliary term it remains to consider is $b-a$. Finally, we obtain

$$\begin{aligned} a^* &= 1-\frac{2\kappa-2}{(2+\delta)\kappa+\delta} \in]0;1[\\ b^* &= a^* \frac{2+2\delta}{2+\delta+\delta\kappa} \in]0;a^*[\end{aligned}$$

  Finally, we obtain the proposed rate. □

Proof of proposition 3. We have seen that for t fix, we control the distance between the characteristic functions of S and σN by a term proportional to $t^2 n^{-c^*}$. Here, t^2 appear because $|t|$ was include in the constants (not depending of n) of the bound of the Lipschitz coefficients. Let Φ be the distribution function of σN and F_n the one of S. Theorem 5.1 p. 142 in Petrov (1995) gives, for every $T>0$:

$$\sup_x |F_n(x)-\Phi(x)| \preceq n^{-c^*}T^2+1/T.$$

We optimize T to obtain the rate of convergence in the central limit theorem. □

4.3 Proof of lemma 1

4.3.1 Existence of Bernoulli shifts with dependent inputs

We first prove the existence of Bernoulli shift with dependent innovations in $\mathbb{L}^1$. The same proof leads to the existence in $\mathbb{L}^m$ for all $m\geq 1$ such that $lm+1\leq m'$. Here we set $\xi^{(s)}=(\xi_{-i}\mathbb{1}_{|i|<s})_{i\in\mathbb{Z}}$ and $\xi_+^{(s)}=(\xi_{-i}\mathbb{1}_{-s<i\leq s})_{i\in\mathbb{Z}}$ for $i\in\mathbb{Z}\cup\{\infty\}$. In order to prove

the existence of Bernoulli shift with dependent innovations, we show that $H(\xi^{(\infty)})$ is the sums of a normally convergent series in $\mathbb{L}^1$. Then formally

$$X_0 = H(\xi^{(\infty)}) = H(0) + (H(\xi^{(1)}) - H(0)) \\ + \sum_{s=1}^{\infty} \Big((H(\xi^{(s+1)}) - H(\xi_+^{(s)})) + (H(\xi_+^{(s)}) - H(\xi^{(s)})) \Big)$$

With (3) we obtain

$$\begin{aligned} |H(\xi^{(1)}) - H(0)| &\leq b_0|\xi_0| \\ |H(\xi^{(s+1)}) - H(\xi_+^{(s)})| &\leq b_{-s}(\|\xi_+^{(s)}\|_\infty^l \vee 1)|\xi_{-s}| \\ |H(\xi_+^{(s)}) - H(\xi^{(s)}| &\leq b_s(\|\xi^{(s)}\|_\infty^l \vee 1)|\xi_s| \end{aligned}$$

Use Hölder inequality yields

$$\mathbb{E}\left|H(\xi^{(1)}) - H(0)\right| + \sum_{s=1}^{\infty} \mathbb{E}\left|H(\xi^{(s+1)}) - H(\xi_+^{(s)})\right| + \mathbb{E}\left|H(\xi_+^{(s)}) - H(\xi^{(s)})\right| \\ \leq \sum_{i\in\mathbb{Z}} 2|i|b_i(\|\xi_0\|_1 + \|\xi_0\|_{l+1}^{l+1}) \quad (18)$$

Hence assumption $l+1 \leq m'$ with $\sum_{i\in\mathbb{Z}} |i|b_i < \infty$ together imply that the variable $H(\xi)$ is well defined. The sam'e way proves that the process $X_n = H(\xi_{n-i}, i \in \mathbb{Z})$ is a well defined process in $\mathbb{L}^1$ and it is strongly stationary. We can extend this result in $\mathbb{L}^m$ for all $m \geq 1$ such that $lm + 1 \leq m'$.

4.3.2 Weakl dependence properties of the model.

We now have to exhibit Lipschitz function and then truncate. We write $\overline{\xi} = \xi \vee (-T) \wedge T$ for a truncation T we will fix further. Then we denote $X_n^{(r)} = H(\xi^{(r)})$ and $\overline{X}_n^{(r)} = H(\overline{\xi}^{(r)})$. Furthermore, for any $k \geq 0$ and any $(u+v)$-tuples such that $s_1 < \cdots < s_u \leq s_u + k \leq t_1 < \cdots < t_v$, we set $X_{\mathbf{s}} = (X_{s_1}, \ldots, X_{s_u})$, $X_{\mathbf{t}} = (X_{t_1}, \ldots, X_{t_v})$ and $\overline{X}_{\mathbf{s}}^{(r)} = (\overline{X}_{s_1}^{(r)}, \ldots, \overline{X}_{s_u}^{(r)})$, $\overline{X}_{\mathbf{t}}^{(r)} = (\overline{X}_{t_1}^{(r)}, \ldots, \overline{X}_{t_v}^{(r)})$. Then we have for all f, g satisfying $\|f\|_\infty, \|g\|_\infty \leq 1$ and $\operatorname{Lip} f + \operatorname{Lip} g < \infty$:

$$\begin{aligned} |\mathrm{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| &\leq |\mathrm{Cov}(f(X_{\mathbf{s}}) - f(\overline{X}_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}))| && (19) \\ &+ |\mathrm{Cov}(f(\overline{X}_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}) - g(\overline{X}_{\mathbf{t}}^{(r)}))| && (20) \\ &+ |\mathrm{Cov}(f(\overline{X}_{\mathbf{s}}^{(r)}), g(\overline{X}_{\mathbf{t}}^{(r)}))|. && (21) \end{aligned}$$

Using $\|g\|_\infty \leq 1$, the term (19) in the sum is bounded with:

$$2\mathrm{Lip}\, f \cdot \mathbb{E}\left|\sum_{i=1}^{u}(X_{s_i} - \overline{X}_{s_i}^{(r)})\right| \leq 2u\mathrm{Lip}\, f\left(\max_{1\leq i\leq u} \mathbb{E}\left|X_{s_i} - X_{s_i}^{(r)}\right| + \max_{1\leq i\leq u} \mathbb{E}\left|X_{s_i}^{(r)} - \overline{X}_{s_i}^{(r)}\right|\right).$$

With the same arguments that for the proof of the existence of $H(\xi^{(\infty)})$ (see equation (18)), the first term in the right sight is bounded with $\sum_{i\geq s} 2|i|b_i(\|\xi_0\|_1 + \|\xi_0\|_{l+1}^{l+1})$. Notice now that if x, y are sequences with $x_i = y_i = 0$ if $|i| \geq r$ then a repeated application of the previous inequality (3) yields

$$|H(x) - H(y)| \leq L(\|x\|_\infty^l \vee \|y\|_\infty^l \vee 1)\|x - y\| \tag{22}$$

where $L = \sum_{i\in\mathbb{Z}} b_i < \infty$ because $\sum_{i\in\mathbb{Z}} |i| b_i < \infty$. Then the second term is bounded using the equation (22):

$$\begin{aligned}
\mathbb{E}\left|X_{s_i}^{(r)} - \overline{X}_{s_i}^{(r)}\right| &= \mathbb{E}\left|H\left(\xi^{(r)}\right) - H\left(\overline{\xi}^{(r)}\right)\right| \\
&\leq L\mathbb{E}\left(\left(\max_{-r\leq i\leq r} |\xi_i|\right)^l \sum_{-r\leq j\leq r} \{|\xi_j| \mathbb{1}_{\xi_j \geq T}\}\right) \\
&\leq L(2r+1)^2 \mathbb{E}\left(\max_{-r\leq i,j\leq r} |\xi_i|^l \{|\xi_j| \mathbb{1}_{|\xi_j|\geq T}\}\right) \\
&\leq L(2r+1)^2 \|\xi_0\|_{m'}^{m'} T^{l+1-m'}
\end{aligned}$$

The second term (20) of the sum is bounded analogously. We write the third term as

$$\left|\mathrm{Cov}(\overline{F}^{(r)}(\xi_{s_i+j}, 1 \leq i \leq u, |j| \leq r), \overline{G}^{(r)}(\xi_{t_i+j}, 1 \leq i \leq v, |j| \leq r)\right|,$$

where $\overline{F}^{(r)} : \mathbb{R}^{u(2r+1)} \to \mathbb{R}$ and $\overline{G}^{(r)} : \mathbb{R}^{u(2r+1)} \to \mathbb{R}$. Under the assumption $r \leq [k/2]$, we use the $\epsilon = \eta$ or λ-weak dependence of ξ in order to bound this covariance term by $\psi(\mathrm{Lip}\,\overline{F}^{(r)}, \mathrm{Lip}\,\overline{G}^{(r)}, u(2r+1), v(2r+1))\epsilon_{k-2r}$, with respectively $\psi(u, v, a, b) = uvab$ or $\psi(u, v, a, b) = uvab + ua + vb$. Let $x = (x_1, \ldots, x_u)$ and $y = (y_1, \ldots, y_u)$ where $x_i, y_i \in \mathbb{R}^{2r+1}$, we bound $\mathrm{Lip}\,\overline{F}^{(r)}$:

$$\mathrm{Lip}\,\overline{F}^{(r)} = \sup \frac{|f(H(\overline{x}_{s_i+l}, 1 \leq i \leq u, |l| \leq r) - f(H(\overline{y}_{s_i+l}, 1 \leq i \leq u, |l| \leq r)|}{\sum_{j=1}^u \|x_j - y_j\|}.$$

Using (22), we have:

$$\begin{aligned}
|\overline{F}^{(r)}(x) - \overline{F}^{(r)}(y)| &\leq \mathrm{Lip}\, f L \sum_{i=1}^u \left(\|\overline{x}_{s_i}\|_\infty \vee \|\overline{y}_{s_i}\|_\infty \vee 1\right)^l \|\overline{x}_{s_i} - \overline{y}_{s_i}\| \\
&\leq \mathrm{Lip}\, f L T^l \sum_{i=1}^u \sum_{-r\leq l\leq r} |x_{s_i+l} - y_{s_i+l}|.
\end{aligned}$$

We thus obtain $\mathrm{Lip}\, F^{(r)} \leq \mathrm{Lip}\, f \cdot L \cdot T^l$. Similarly $\mathrm{Lip}\, G^{(r)} \leq \mathrm{Lip}\, g \cdot L \cdot T^l$. In the η-weak dependence, we bound the covariance:

$$\begin{aligned}
&|\mathrm{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leq (u\mathrm{Lip}\, f + v\mathrm{Lip}\, g) \\
&\left[4\sum_{i\geq r} |i| b_i (\|\xi_0\|_1 + \|\xi_0\|_{l+1}^{l+1}) + (2r+1)L\left((2r+1)2\|\xi_0\|_{m'}^{m'} T^{l+1-m'} + T^l \eta_{\xi,k-2r}\right)\right]
\end{aligned}$$

We then fix the truncation $T^{m'-1} = \dfrac{2(2r+1)\|\xi_0\|_{m'}^{m'}}{\eta_{\xi,k-2r}}$ to obtain the result of the lemma 1 in the η-weak dependent case. Under the λ-weak dependence:

$$
\begin{aligned}
&|\mathrm{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \le (u\mathrm{Lip}\, f + v\mathrm{Lip}\, g + uv\mathrm{Lip}\, f\mathrm{Lip}\, g)\\
&\quad \left(\left\{4\sum_{i\ge r}|i|b_i(\|\xi_0\|_1 + \|\xi_0\|_{l+1}^{l+1}) + (2r+1)L\left(2(2r+1)T^{l+1-m'}\|\xi_0\|_{m'}^{m'} + T^l\lambda_{\xi,k-2r}\right)\right\}\right.\\
&\qquad\qquad\qquad\qquad \left.\vee\ \left\{(2r+1)^2L^2T^{2l}\lambda_{\xi,k-2r}\right\}\right)
\end{aligned}
$$

We then fix the truncation $T^{l+m'-1} = \dfrac{2\|\xi_0\|_{m'}^{m'}}{L\lambda_{\xi,k-2r}}$ to obtain the result of the lemma 1 in the η-weak dependent case.

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