Renormalization of the local time for the $d$-dimensional fractional Brownian motion with $N$ parameters.

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Abstract

We study the asymptotic behavior in Sobolev norm of the local time of the $d$–dimensional fractional Brownian motion with $N$–parameters when the space variable tends to zero, both for the fixed time case and when simultaneously time tends to infinity and space variable to zero.

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1 Introduction

Let $B^H = \{B^H_t : t \geq 0\}$ be a standard fractional Brownian motion (fBm for brevity) with Hurst parameter $H \in (0,1)$. It is well known that this process is a centered Gaussian process which admits an integral representation of the form

$$B^H_t = \int_0^t K^H_H(t,s)dW_s,$$

where $W$ is an standard Wiener process. The kernel $K^H_H(t,s)$ is given, for $s < t$, by

$$K^H_H(t,s) = c^H_H(t-s)^\mu - \mu c^H_H \int_s^t (r-s)^{\mu-1}(1-(\frac{s}{r})^{-\mu})dr,$$  \hspace{1cm} (1)

with $c^H_H$ being a constant and $\mu = H - \frac{1}{2}$.

The covariance function of $B^H_t$ can be represented as

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\[ R_H(s, t) = \mathbb{E}(B^H_s B^H_t) = \int_0^{s \land t} K_H(t, r) K_H(s, r) dr, \]

and has the explicit form
\[ R_H(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}). \]

A very good survey about the fBm is the paper of Nualart [5].

For \( H = (H_1, \ldots, H_N) \) the \((N, 1)\)-fBm is defined as
\[ B^H_t = \int_{[0,t]} K^H(t, s) dW_s, \]

where \( K^H(t, s) = \bigotimes_{j=1}^N K_{H_j}(t_j, s_j) \), \( s, t \in \mathbb{R}_+^N \) and \( W \) is an standard \( N \)-parameter Brownian motion. Its covariance function is
\[ R^H_H(s, t) = \mathbb{E}(B^H_s B^H_t) = \prod_{j=1}^N R_{H_j}(s_j, t_j). \]

Finally given the \( N \times d \)-matrix \( \mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d) \) where for \( i = 1, \ldots, d \) and \( j = 1, \ldots, N \), \( \mathcal{H}_i = (H_{i,1}, \ldots, H_{i,N}) \) is a column vector and \( H_{i,j} \in (0, 1) \), the \( N \)-parameter, \( d \)-dimensional fractional Brownian motion \((N, d)\)-fBm for brevity) is defined by
\[ B^\mathcal{H}_t = (B^\mathcal{H}_1, \ldots, B^\mathcal{H}_d)_{t \in \mathbb{R}_+^N} \]

where its components are independent and for every \( i = 1, \ldots, d \), \( B^\mathcal{H}_i \) is a \((N, 1)\)-fBm with Hurst parameter \( \mathcal{H}_i \).

For any \( t \in \mathbb{R}_+^N \) and \( x \in \mathbb{R}^d \), the local time \( L(t, x) \) of the \((N, d)\)-fBm can be defined as the density of the occupation measure \( \mu_t \), defined as
\[ \mu_t(A) = \int_{[0,t]} 1_A(B^\mathcal{H}_s) ds, \quad A \in \mathcal{B}(\mathbb{R}^d). \]

Formally, we can write
\[ L(t, x) = \int_{[0,t]} \delta_x(B^\mathcal{H}_s) ds \]

where \( \delta_x \) denotes the Dirac function and \( \delta_x(B^\mathcal{H}_s) \) is therefore a distribution in the Watanabe sense (see [6]).

This local time for \((N, d)\)-fBm has been studied by Xiao and Zhang [7], Hu and Oksendal [2] and Eddahbi et al. [1] between others.

The aim of this paper is to study the asymptotic behavior of \( L(t, x) \) when \(|x| \), the euclidean norm of \( x \) in \( \mathbb{R}^d \), goes to 0, both for a fixed time and when the time goes to infinity, and we renormalize his Sobolev norm. We generalize the results of [3] from the \((N, d)\)-standard Brownian motion to the \((N, d)\)-fractional Brownian motion. In the standard Brownian motion case, the covariance function is simply \( R_2(s, t) = s \land t \). Here, the control of the covariance function \( R_H(s, t) \) for \( H \neq \frac{1}{2} \) is the main difficulty.

Section 2 is devoted to the presentation of the problem. In particular we review from [1] the chaotic decomposition of the local time \( L(t, x) \) as a functional of the
(N, d)–fBm and its regularity in terms of Sobolev–Watanabe norms. In section 3 we present a list of auxiliary lemmas. Section 4 is devoted to the presentation and proof of the main result, namely the asymptotic behavior of this local time, for fixed t, in the case \( H_{i,j} = H, \forall i, j \), when \(|x|\) goes to 0. In section 5 we extend the result to the case \( t := t_1 \cdots t_N \) going to infinity.

2 Preliminaries and statement of the problem

If \( F \) is a square integrable Brownian random variable, it can be represented by its Wiener chaos expansion

\[
F = \sum_{n=0}^{\infty} I_n(f_n),
\]

where \( I_n(f_n) \) denotes the multiple Itô stochastic integral of the symmetric kernel \( f_n \in L^2(\mathbb{R}^d_+) \) with respect to the Wiener process \( W \).

If \( L \) is the Ornstein–Uhlenbeck operator

\[
LF = -\sum_{n=0}^{\infty} nI_n(f_n),
\]

\( p \in (1, \infty) \) and \( \alpha \in \mathbb{R} \), we define the Sobolev–Watanabe spaces \( \mathbb{D}^{\alpha,p} \) as the closure of the set of polynomial random variables with respect to the norm

\[
\|F\|_{\alpha,p} = \|(\text{Id} - L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},
\]

where \( \text{Id} \) stands for the identity mapping.

We denote by \( D \) the chaotic derivative operator. It acts on multiple Itô stochastic integrals as

\[
D_t(I_n(f_n)) = nI_{n-1}(f_n(\cdot, t)),
\]

and is continuous from \( \mathbb{D}^{\alpha,p} \) into \( \mathbb{D}^{\alpha-1,p}(L^2(\mathbb{R}^d_+)) \).

It is known that a Brownian random variable \( F \) belongs to \( \mathbb{D}^{\alpha,2} \) if and only if its chaotic decomposition \( \sum_{n=0}^{\infty} I_n(f_n) \) satisfies

\[
\sum_{n=0}^{\infty} (1 + n)^\alpha \|I_n(f_n)\|_2^2 < \infty,
\]

where \( \|I_n(f_n)\|_2^2 = n!\|f_n\|_2^2 \).

Set \( \mathbb{D}^{\infty,2} = \bigcap_{\alpha \in \mathbb{R}} \mathbb{D}^{\alpha,2} \). If \( F \in \mathbb{D}^{\infty,2} \), we can compute its chaos expansion using the Stroock formula

\[
F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E}(D^n F)).
\]

For a complete survey of this subjects we refer the reader to the book of Watanabe [6].
Let $p_\varepsilon(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. Consider also, for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\varepsilon^d(x) = \prod_{i=1}^d p_\varepsilon(x_i), \ x = (x_1, \ldots, x_d).$$

We denote by $H_n$ the $n$–th Hermite polynomial, defined for $n \geq 1$, by

$$H_n(x) = \left(-\frac{1}{n!}\right)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right)\right), \ x \in \mathbb{R}$$

and $H_0(x) = 1$.

As we proved in [1] the chaotic decomposition of the local time of the $(N, d)$–fBm is

$$L(t, x) = \sum_{n_1, \ldots, n_d \geq 0} \int_{[0,t]} \prod_{i=1}^d p_{\frac{x_i}{s_{ni}}} \left(\frac{x_i}{s_{ni}}\right) H_{n_i} \left(K_{\frac{\pi_{s_i}}{s_{ni}}}(s, \cdot)^{s_{ni}}\right) ds,$$

provided that $\sum_{j=1}^N \frac{1}{H^*_j} > d$, where $t \in \mathbb{R}^+, x \in \mathbb{R}^d, s = s_1 \cdots s_N$ and $s_{Hi} = \prod_{j=1}^N s_{Hi,j}$. The integrals $I_{n_i}^i$ denotes the multiple Itô stochastic integrals with respect to the independent $N$–parameter Wiener processes $W^i$. Finally $H^*_j = \max\{H_{i,j}, i = 1, \ldots, d\}$.

Moreover, in [1] we proved that this functional belongs to the space $\mathbb{D}^{\alpha,2}$ if

$$\alpha < \sum_{j=1}^N \frac{1}{2H^*_j} - \frac{d}{2}$$

If all $H_{i,j} = H$, this expression becomes $\alpha < \frac{N}{2H} - \frac{d}{2}$, and then a sufficient condition for the local time to be in $L^2(\Omega)$ is $N > Hd$. Observe that this sufficient condition is also founded in Xiao and Zhang [7]. From now on we will suppose always this condition.

Recall that if $H = \frac{1}{2}$,

$$\sum_{j=1}^N \frac{1}{2H^*_j} - \frac{d}{2} = N - \frac{d}{2},$$

which is the same condition obtained in [3] for the $N$–parameter Wiener process in $\mathbb{R}^d$.

3 Auxiliary lemmas

Lemma 1 If $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ we have

$$\sup_{x \in \mathbb{R}} |\sqrt{n!H_n(x)} e^{-\beta x^2}| \leq c(n \lor 1)^{-\frac{8\beta - 1}{2\beta}}$$
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Proof: This result is proved in [4]. □

Remark 2 The factor $\sqrt{n!}$ appears because we do not use the same definition of Hermite polynomials as in [4].

Lemma 3 Let $d \geq 1$ and $\nu \in (0, 1)$. We can choose a universal constant $c$ such that for any $m \geq 1$,

$$\sum_{n_1 + \cdots + n_d = m} \prod_{i=1}^{d} (n_i \lor 1)^{-\nu} \leq cm^{d(1-\nu)-1}$$

Proof: This result is proved in [4]. □.

Lemma 4 Let $\gamma$ and $a$ be positive constants and $b \in \mathbb{R}$. Set $\alpha = \frac{b-1}{a}$. Then

$$\int_{[0,1]^N} \exp(-\gamma s^a) \frac{ds}{s^b} = \frac{1}{(N-1)!} \left( \frac{1}{a} \right)^N \gamma^{-a} g_{N-1}(\gamma, \alpha)$$

where

$$g_{N-1}(\gamma, \alpha) := \int_{\gamma}^{\infty} t^{a-1} e^{-t} (\log t) t^{-1} dt.$$

Proof: Using the change of variables $u_1 = s_1 \cdots s_N, u_2 = s_2 \cdots s_N, \ldots, u_N = s_N$, with Jacobi determinant $\frac{1}{u_2 - u_N}$, we obtain

$$\int_{[0,1]^N} \exp(-\gamma s^a) \frac{ds}{s^b} = \int_{\{0 \leq u_1 \leq \cdots \leq u_N \leq 1\}} \frac{1}{u_1^a} \exp(-\gamma \frac{du_N \cdots du_2}{u_N \cdots u_2}) du_1$$

$$= \frac{1}{(N-1)!} \int_0^1 \left( \frac{1}{r} \right)^{N-1-1} \frac{1}{r^b} \exp(-\frac{\gamma}{r^a}) dr,$$

and making the change of variable $\gamma r^{-a} = t$ we get the desired result. □.

Lemma 5 The function

$$Q_H(z) = \begin{cases} \frac{R_H(1, z)}{z^H} & \text{if } z \in (0, 1] \\ 0 & \text{if } z = 0, \end{cases}$$

has the following properties:

1. It is strictly increasing and it continously maps $[0, 1]$ onto $[0, 1]$. Moreover, $Q_H(1) = 1$.

2. For fixed $\delta \in (0, 1)$ and for any $z \in [0, 1 - \delta]$, it satisfies the inequality

$$Q_H(z) \leq c(H, \delta) z^G,$$

where $G = H \land (1 - H)$. 
3. For fixed $\delta \in (0, 1)$ and $\beta > 0$, it satisfies the inequality

$$\int_{1-\delta}^{1} Q_H(z)^\beta dz \leq \frac{c(H, \delta)}{\beta^{1/\gamma}}.$$ 

**Proof:**

The proof of parts 1 and 3 are done in [1].

For the part 2 we have

$$Q_H(z) = 1 - (1 - z)^{2H}.$$ 

Using Taylor expansion, $1 - (1 - z)^{2H} = 2H(1 - \theta)^{2H-1}z$ with $0 \leq \theta \leq z$.

If $H \geq \frac{1}{2}$, we have $1 - (1 - z)^{2H} \leq 2Hz$, and therefore $Q_H(z) \leq Hz^{1-H} + \frac{1}{2}z^H \leq c_1z^{1-H}$, $c_1$ being a positive constant.

If $H < \frac{1}{2}$ and $z \in [0, 1 - \delta]$, we have $1 - (1 - z)^{2H} \leq 2Hz\delta^{2H-1}z$, and then $Q_H(z) \leq Hz\delta^{2H-1}z^{1-H} + \frac{1}{2}z^H \leq c_2z^H$, $c_2$ being another positive constant. □

In what follows, for every $x > 0$ and $\gamma \geq 0$, we denote the complementary incomplete Gamma function as

$$\Gamma(x, \gamma) = \int_{\gamma}^{\infty} e^{-t}t^{x-1}dt.$$ 

In particular $\Gamma(x) := \Gamma(x, 0)$ and $\Gamma(x, \gamma) \leq \Gamma(x)$.

**Lemma 6** The function

$$g_{N-1}(\gamma, \alpha) := \int_{\gamma}^{\infty} t^{N-1}e^{-t}(\log \frac{t}{\gamma})^{N-1}dt.$$ 

has the following behavior when $\gamma$ tends to 0:

1. If $\alpha > 0$, $g_{N-1}(\gamma, \alpha) = (\log \frac{1}{\gamma})^{N-1}\Gamma(\gamma, \alpha) + \mathcal{O}((\log \frac{1}{\gamma})^{N-2}).$

2. If $\alpha = 0$, $g_{N-1}(\gamma, \alpha) = e^{-\gamma} \frac{1}{N}(\log \frac{1}{\gamma})^N + \mathcal{O}((\log \frac{1}{\gamma})^{N-1}).$

3. If $\alpha < 0$, $g_{N-1}(\gamma, \alpha) = \gamma^\alpha(\frac{\Gamma(N)}{\log N} + o(\gamma)).$

**Proof:**

Note first that

$$(\log \frac{t}{\gamma})^{N-1} = \sum_{k=0}^{N-1} \binom{N-1}{k} (\log \frac{1}{\gamma})^{N-1-k}(\log t)^k. \quad (2)$$

Then,
• If $\alpha > 0$, the function 
  \[ t \mapsto t^{\alpha - 1} e^{-t (\log t)^k} \]
  is always integrable on $[0, \infty)$ for any $k \in \mathbb{N}$. Therefore,
  \[ g_{N-1}(\gamma, \alpha) = (\log \frac{1}{\gamma})^{N-1} \Gamma(\gamma, \alpha) + O((\log \frac{1}{\gamma})^{N-2}). \]

• If $\alpha = 0$, we need to estimate the integral
  \[ g_{N-1}(\gamma, 0) = \int_{\gamma}^{\infty} t^{-1} e^{-t (\log \frac{t}{\gamma})^N} dt. \]
  Integrating by parts we obtain
  \[ g_{N-1}(\gamma, 0) = \frac{1}{N} \int_{\gamma}^{\infty} e^{-t (\log \frac{t}{\gamma})^N} dt = \frac{e^{-\gamma}}{N} (\log \frac{1}{\gamma})^N + O((\log \frac{1}{\gamma})^{N-1}) \]

• If $\alpha < 0$, making the change of variable $s = -\alpha \log(\frac{t}{\gamma})$, the result follows immediately.

\[ \square \]

4 Renormalization of the local time for fixed $t$

The main purpose of this section is to study the asymptotic behavior of $L(t, x)$, for $t \in \mathbb{R}^N$ and $x \in \mathbb{R}^d$, when $|x| \to 0$. In the case $dH \geq 1$ it has a singularity. An interesting question is to renormalize the local time, that means, to find a deterministic function $f(t, x)$ such that $f(t, x)L(t, x)$ converge to 1 in some precise sense. We will do it with respect the norm $\| \cdot \|_{\alpha, 2}$. Then we will obtain a function $f(t, x)$ such that $\|f(t, x)L(t, x)\|_{\alpha, 2}$ converges to 1 when $|x| \to 0$, both for fixed $t$ and when $\underline{t} = t_1 \cdots t_N \to \infty$.

Recall the expression of the $\mathbb{D}^{\alpha, 2}$-norm of the local time $L(t, x)$. For the sake of simplicity we take $t := (1, \ldots, 1)$.

We have
\[ \|L(\underline{1}, x)\|_{\alpha, 2}^2 = \sum_{m=0}^{\infty} (1 + m)^\alpha A_m(x), \tag{3} \]
where
\[ A_m(x) = \sum_{n_1 + \cdots + n_d = m} \left\| \int_{[0,1]} \left( \prod_{i=1}^{d} \frac{p_{\frac{x_i}{\underline{n}_i}}(x_i)}{\underline{n}_i!} \mathbb{H}_{n_i}(\frac{x_i}{\underline{n}_i}) \right) I_{n_i}(K_{\underline{n}_i}(s, \cdot) \otimes n_i) \right\|_{L^2(\Omega)}^2, \]
and as

\[ E(I_n^K(K_{H_i}(u, \cdot)^{\otimes n_i})I_n^K(K_{H_j}(v, \cdot)^{\otimes n_j})) = \delta_{ij} n_i! \left( R_{H_i}(u, v) \right)^{n_i}, \]

\[ A_m(x) = \sum_{n_1 + \cdots + n_d = m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^d \left( \frac{R_{H_{i,j}}(u_{i,j}, v_{i,j})}{(u_{i,j})^{H_{i,j}}} \right)^{n_i} \]

\[ \times n_i! H_{n_i}(\frac{x_i}{\pi H_i})H_{n_i}(\frac{x_i}{\pi H_i})p_{2n}(x_i)p_{2n}(x_i), \]

and in particular

\[ A_0(x) = \left( \int_{[0,1]^N} ds \prod_{i=1}^d \frac{1}{(2\pi)^{2H_i} s_{i,j}^{2H_i-j}} \exp\left( - \frac{x_i^2}{2s_{i,j}^{2H_i-j}} \right) \right)^2. \]

In all this section we confine our attention to the situation where \( H_{i,j} = H \) for all \( (i,j) \in \{1, \ldots, d\} \times \{1, \ldots, N\} \), and use the notation \( B^H \) for \( B^H \).

Observe that in this particular case

\[ A_0(x) = \frac{1}{(2\pi)^d \gamma} \left( \int_{[0,1]^N} ds \prod_{i=1}^d \frac{1}{\gamma s_{i,j}^{2H_i-j}} \exp\left( - \frac{x_i^2}{2s_{i,j}^{2H_i-j}} \right) ds \right)^2, \]

and

\[ A_m(x) = \sum_{n_1 + \cdots + n_d = m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^d \left( \frac{R_H(u_{i,j}, v_{i,j})}{(u_{i,j})^{H_{i,j}}} \right)^{n_i} \]

\[ \times n_i! H_{n_i}(\frac{x_i}{\pi H})H_{n_i}(\frac{x_i}{\pi H})p_{2n}(x_i)p_{2n}(x_i)du dv. \]

Our main result is the following:

**Theorem 7** Let \( B^H \) be \( (N, d) \)-fBm. Set \( \lambda := \frac{d}{H}. \) For any \( \alpha < \frac{N}{2H} - \frac{d}{2} \) we have:

1) If \( \lambda > 0 \), \( \lim_{|x| \to 0} \|L(1, x)\|_{\alpha, 2} = \frac{2^\frac{\lambda}{2}(\pi H)^N|x|^{-\lambda}}{(2\pi)^N N!} \left( \log \frac{2}{|x|} \right)^{N-1} \Gamma\left( \frac{\lambda}{2} \right)^{-1} = 1. \)

2) If \( \lambda = 0 \), \( \lim_{|x| \to 0} \|L(1, x)\|_{\alpha, 2} = \frac{1}{(2\pi)^N N!} \left( \log \frac{2}{|x|} \right)^{N-1} = 1. \)

3) If \( \lambda < 0 \),

\[ \lim_{|x| \to 0} \|L(1, x)\|_{\alpha, 2} = \|L(1, 0)\|_{\alpha, 2} = \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{1 - H\lambda} \right)^{\frac{N}{2}} \]

\[ \times \left[ \sum_{r=0}^{\infty} (1 + 2r)^\alpha \prod_{r_1 + \cdots + r_d = r} \left( \frac{(2r_i)!}{r_i!} \right)^{\frac{d}{2}} \left( \int_0^1 Q_H(z) \frac{dz}{z^{dH}} \right)^{\frac{N}{2}} \right] < \infty. \]
Remark 8 This theorem shows that for $\lambda \geq 0$ the local time explodes at the origin and for $\lambda < 0$ it does not. Observe that if $H = \frac{1}{2}$, we have that the local time explodes at the origin if and only if $d \geq 2$, as is discussed in [3].

Proof:

The idea of the proof is to show that the convergence of $A_m(x)$ for any $m \geq 1$ when $|x| \to 0$, is controlled by $A_0(x)$ and then the asymptotic behavior of $L(1, x)$ coincides with the asymptotic behavior of $A_0(x)^{\frac{1}{2}}$.

Define now, for $\gamma > 0$ and $m \geq 0$,

$$B_m(\gamma) = \int_{[0,1]^N} \int_{[0,1]^N} \frac{(\prod_{j=1}^{N} R_H(u_j, v_j))^{m}}{(u \cdot v)^{H(m+d)}} \exp(-\frac{\gamma}{u^{2H}}) \exp(-\frac{\gamma}{v^{2H}}) dudv.$$

Clearly,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0(\frac{1}{2}|x|^2).$$

For $m \geq 1$, choosing $\beta \in [\frac{1}{4}, \frac{1}{2}]$, we can write

$$A_m(x) = \sum_{n_1+\ldots+n_\alpha = m} \int_{[0,1]^N} \int_{[0,1]^N} (\prod_{j=1}^{N} R_H(u_j, v_j)^{m}) \frac{1}{(u \cdot v)^{\alpha H}}$$

$$\times \prod_{i=1}^{d} \sqrt{n_i!} H_{n_i} \left( \frac{x_i}{\sqrt{u_H}} \right) \exp\{-\beta \frac{x_i^2}{2u_H}\} \sqrt{n_i!} H_{n_i} \left( \frac{x_i}{\sqrt{v_H}} \right) \exp\{-\beta \frac{x_i^2}{2v_H}\}$$

$$\times \exp\{-\frac{1}{2} - \beta \} \frac{x_i^2}{2u_H}\} \exp\{-\frac{1}{2} - \beta \} \frac{x_i^2}{2v_H}\} dudv,$$

and applying Lemmas 1 and 2 we obtain

$$A_m(x) \leq c \frac{1}{(2\pi)^d} m^{d(1 - \frac{\alpha}{2d}) - 1} B_m((\frac{1}{2} - \beta)|x|^2).$$

Then our problem reduces to the study of the asymptotic behavior of $B_m$.

As $R_H(u_j, v_j) = R_H(1, \frac{v_j}{u_j})u_j^{2H}$, we have

$$B_m(\gamma) = 2^N \int_{[0,1]^N} \int_{[0,1]^N} \cdots \int_{[0,1]^N} \prod_{j=1}^{N} R_H(1, \frac{v_j}{u_j})^{m} u_j^{2Hm} \frac{1}{(u_j v_j)^{H(m+d)}} \exp(-\frac{\gamma}{u^{2H}}) \exp(-\frac{\gamma}{v^{2H}}) dv_1 \cdots dv_N.$$ 

With the change $\frac{\gamma}{u_j} = z_j, \forall j = 1, \ldots, N$ and computing iteratively the previous integral, we find

$$B_m(\gamma) = 2^N \int_{[0,1]^N} \int_{[0,1]^N} \cdots \int_{[0,1]^N} \frac{1}{z_j^{H(m+d)}} \exp(-\frac{\gamma z_j}{2H}) dz_1 \cdots dz_N.$$
where \( \kappa(r) = 1 + \frac{1}{2\pi r} \).

By Lemma 4, with \( a = 2H \) and \( b = 2Hd - 1 \), we have

\[
J_N(\gamma, z) = \int_{[0,1]^N} u^{1-2Hd} \exp\left(\frac{-\kappa(z)\gamma}{u^{2H}}\right) du = c(N, d, H)\gamma^{-\lambda} \int_\gamma^\infty e^{-s\kappa(z)\gamma} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} ds,
\]

where \( \lambda = d - \frac{1}{H} = \frac{b-1}{a} \).

Therefore

\[
B_m(\gamma) = \int_{[0,1]^N} \prod_{j=1}^N R_H(1, z_j)^m \cdot e^{\frac{-z_j}{z(Hm)}} e^{-s\kappa(z)\gamma} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} dz ds.
\]

First we will see that for \( m > \frac{H}{G} \), we have

\[
B_m(\gamma) \leq c(H, d, N)\gamma^{-\lambda} g_{N-1}(\gamma, \lambda) m^{-\frac{N}{2\pi}}.
\]

Indeed, controlling the exponential by 1, we obtain

\[
B_m(\gamma) \leq c(N, H, d)\gamma^{-\lambda} \int_\gamma^\infty \prod_{j=1}^N \int_{[0,1]^N} R_H(1, z_j)^m \cdot e^{\frac{-z_j}{z(Hm+d)}} e^{-s\kappa(z)\gamma} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} dz ds
\]

\[
= c(N, H, d)\gamma^{-\lambda} g_{N-1}(\gamma, \lambda) \left( \int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz \right)^N,
\]

where the function \( Q_H \) is introduced in lemma 5.

Now, choosing \( \delta \in (0, 1) \), we have

\[
\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz \leq \int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz + (1-\delta)^{-dH} \int_{1-\delta}^1 Q_H(z)^m dz
\]

The second summand on the right, using part 3 of Lemma 4, is controlled by \( c(H, \delta)m^{-\frac{N}{2\pi}} \).

For the first summand, if \( m > \frac{dH-1}{\alpha} = \frac{H}{G} \), we fix \( \alpha \in \left( \frac{H}{G}, m \right) \), and write

\[
\int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz = \int_0^{1-\delta} Q_H(z)^{m-\alpha} Q_H(z)^\alpha \frac{1}{z^{dH}} dz.
\]

Using that \( Q_H \) is an increasing function and part 2 of Lemma 5, we control this by

\[
Q_H(1-\delta)^{m-\alpha} c(H, \delta, \alpha) \int_0^{1-\delta} z^{\alpha G-dH} dz.
\]

As \( \alpha > \frac{H}{G} \), the integral that appears in the last expression is a constant that depends on \( H, d, \alpha \) and \( \delta \).
Therefore, being $Q_H(1 - \delta) < 1$, we can estimate this term by
\[ c(H, d, \delta, \alpha)m^{-\frac{1}{2N}}, \]
and we get (4).

Note that this result is true only for $m > \frac{\lambda H}{G}$. If $\lambda \leq 0$ this covers all cases. But if $\lambda > 0$ the $B_m$ terms with $m \leq \frac{\lambda H}{G}$ are not controlled yet. The following part of the proof will discuss these first terms.

Observe that for any $0 < \epsilon < m$, being $Q_H(1 - \delta) < 1$, we can estimate this term by
\[ c(H, d, \delta, \alpha) \gamma^{-\lambda} g_{N-1}(\gamma, \alpha) \]
where $\alpha$ is some positive constant depending also on $\epsilon$.

Indeed, putting $c = c(N, H, d)$,
\[
B_\epsilon(\gamma) = c_{\gamma^{-\lambda}} \int_0^\infty \int_0^1 \prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s^2}{2Hd}}}{s^{\lambda}} e^{-s \lambda^{-1}(\log \frac{s}{\gamma})^{N-1}} dzds
\]

because the function
\[
\prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s^2}{2Hd}}}{s^{\lambda}}
\]
is symmetric in $z$.

Now estimating $Q_H$ and the exponential by 1 in the integrals between $1 - \delta$ and 1, we obtain
\[
B_\epsilon(\gamma) \leq c_{\gamma^{-\lambda}} \int_0^\infty \sum_{k=0}^N \left( \begin{array}{c} N \\ k \end{array} \right) \int_0^{1-\delta} \cdots \int_0^{1-\delta} \prod_{j=1}^{N-k} Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s^2}{2Hd}}}{s^{\lambda}} e^{-s \lambda^{-1}(\log \frac{s}{\gamma})^{N-1}} dz_1 \cdots dz_k ds
\]

\[
\leq c_{\gamma^{-\lambda}} \int_0^\infty \sum_{k=0}^N \left( \begin{array}{c} N \\ k \end{array} \right) \int_0^{1-\delta} \cdots \int_0^{1-\delta} \left( \frac{\delta}{(1 - \delta)^d} \right)^{N-k} c(H, \delta) \epsilon
\]
\[
\times \prod_{j=1}^{k} z_j^{e^G-dH} e^{-\frac{s}{(\lambda_1-\epsilon)\pi\Gamma}} e^{-\lambda s^2} (\log \frac{s}{\gamma})^{N-1} \, dz_1 \ldots dz_k ds,
\]

where we have used section 2 of Lemma 5.

Now, choosing \( \epsilon < \frac{dH}{2H} \), we can use Lemma 4 with \( a = 2H, b = -\epsilon G + dH, \gamma = s, N = k \) and \( \alpha = \frac{dH - \epsilon G - 1}{2H} \), to bound the right hand side of the last inequality by

\[
c\gamma^{-\lambda} \int_{\gamma}^\infty \sum_{k=0}^{N} \int_{\gamma}^\infty (\log \frac{t}{s})^{k-1} s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-t} e^{-s} (\log \frac{s}{\gamma})^{N-1} dt ds,
\]

where \( c \) is a constant that depends on \( H, d, N, \epsilon, k, \delta \).

Using the fact that for any \( n \geq 1 \) and for \( t \geq s \)

\[
\log \frac{t}{s} \leq n \left( \frac{t}{s} \right)^{\frac{1}{n}},
\]

and taking \( n = M(k - 1) \) for a big \( M \), we obtain

\[
B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \sum_{k=0}^{N} \int_{\gamma}^\infty \int_{\gamma}^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-t} e^{-s} (\log \frac{s}{\gamma})^{N-1} dt ds,
\]

where \( c \) depends also on \( M \). From now on in each expression \( c \) will be the suitable constant.

As \( \epsilon < m < \frac{\lambda H}{G} \), we have \( \lambda - \frac{\epsilon G}{2H} + \frac{1}{M} > 0 \) and

\[
B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \sum_{k=0}^{N} c \int_{\gamma}^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} \Gamma(s, \frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M})(\log \frac{s}{\gamma})^{N-1} ds.
\]

Controlling the truncated Gamma function by the corresponding Gamma function we obtain

\[
B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \int_{\gamma}^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} (\log \frac{s}{\gamma})^{N-1} ds
\]

\[
= c\gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}).
\]

Observe that for \( M \) sufficiently large

\[
\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.
\]

Finally for the \( m = 0 \) case, using Lemma 4, we have immediately, as \( \alpha = \frac{\lambda}{2} \)

\[
B_0(\gamma) = \frac{1}{((N-1)!)^2 (2H)^{2N}} \gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2}).
\]
Therefore we have to separate the cases $\lambda \geq 0$ and $\lambda < 0$.

For $\lambda \geq 0$ we have

$$\|L(\tilde{1}, x)\|_{\alpha, 2}^2 = \sum_{m=0}^{\infty} (1 + m)\alpha A_m(x)$$

- The terms $A_m$ with $m = 1, \ldots, \lceil \frac{\lambda H}{2\alpha} \rceil$ are controlled by

$$c\gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}) m^{(1 - \frac{8\beta - 1}{6})^{-1}}$$

where $\gamma = (\frac{1}{2} - \beta)|x|^2$, and $\epsilon$ and $M$ satisfy

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$  

Then, by Lemma 6, part 1, this is asymptotically, when $\gamma \downarrow 0$, as

$$c\gamma^{-\lambda} \left( \log \frac{1}{\gamma} \right)^{N-1}$$

- The terms $A_m$ with $m \geq \lceil \frac{\lambda H}{2\alpha} \rceil + 1$ are controlled by

$$cm^{(1 - \frac{8\beta - 1}{6})^{-1}} \gamma^{-\lambda} g_{N-1}(\gamma, \lambda)m^{-\frac{N}{2H}}.$$  

Then

$$\sum_{m > \frac{\lambda H}{2\alpha}} (1 + m)\alpha A_m(x) \leq c \sum_{m > \frac{\lambda H}{2\alpha}} m^{d(1 - \frac{8\beta - 1}{6})^{-1}} m^{-\frac{N}{2H}} (1 + m)^\alpha \gamma^{-\lambda} g_{N-1}(\gamma, \lambda),$$

and using the fact that $\alpha < \frac{N}{2H} - \frac{d}{2}$, we have that the series between keys is convergent and the asymptotic behavior of the last expression is, by lemma 6, as

$$c\gamma^{-\lambda} \left( \log \frac{1}{\gamma} \right)^{N-1}.$$  

Finally,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0 \left( \frac{1}{2} |x|^2 \right) = \frac{1}{(2\pi)^d ((N - 1)!)^2} \frac{1}{(2H)^2N} \gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2})^2,$$

where $\gamma = \frac{|x|^2}{2}$. When $\gamma \downarrow 0$, this goes to $\infty$ as $\gamma^{-\lambda} \left( \log \frac{1}{\gamma} \right)^{2N-2}$, and as the exponent of the logarithm is $2N-2$, this term dominates the asymptotic behavior. Note that we consider $A_0^\frac{\gamma}{2}$ in place of $A_0$, to get the functions that appear in the theorem.
The $\lambda < 0$ case follows directly. As we have seen before,

$$\sum_{m \geq 1} (1 + m)^{\alpha} A_m(x)$$

is controlled by $\gamma^{-\lambda}g_{N-1}(\gamma, \lambda)$, and by Lemma 6, part 3, this term goes to a constant when $\gamma \downarrow 0$.

In this case the norm $\|L(t, x)\|_{\alpha, 2}$ is continuous. Therefore we don’t have an explosion, and

$$\lim_{|x| \to 0} \|L(1, x)\|_{\alpha, 2} = \|L(1, 0)\|_{\alpha, 2} = (\sum_{m=0}^{\infty} (1 + m)^{\alpha} A_m(0))^\frac{1}{2},$$

where

$$A_m(0) = \frac{1}{(2\pi)^d} \sum_{n_1 + \cdots + n_d = m} \prod_{i=1}^{d} n_i! H_{n_i}(0)^2 B_m(0),$$

and

$$B_m(0) = \frac{1}{2^N} \int_{[0,1]^N} \left( \int_{[0,1]^N} u^{1-2Hd} d\mu_1 \cdots d\mu_N \right) \prod_{j=1}^{N} R_{H}(1, z_j)^{m} z_j^{H(m+d)} dz_1 \cdots dz_N.$$

$$= 2^N \left( \int_{0}^{1} u^{1-2Hd} du \right)^{N} \left( \int_{0}^{1} Q_{H}(z)^{m} \frac{dz}{z^dH} \right)^{N}$$

$$= (\frac{1}{1-Hd})^{N} \int_{0}^{1} Q_{H}(z)^{m} \frac{dz}{z^dH}.\frac{dz}{dH}^{N}.$$

Note that as $\lambda < 0$, $1 - 2Hd > -1$.

Finally,

$$\|L(1, 0)\|_{\alpha, 2}^{2} = \frac{1}{(2\pi)^d} \left( \frac{1}{1-Hd} \right)^{N} \sum_{m=0}^{\infty} (1+m)^{\alpha} \left( \sum_{n_1 + \cdots + n_d = m} \prod_{i=1}^{d} n_i! H_{n_i}(0)^2 \right) \left( \int_{0}^{1} Q_{H}(z)^{m} \frac{dz}{z^dH} \right)^{N}$$

$$= \frac{1}{(2\pi)^d} \left( \frac{1}{1-Hd} \right)^{N} \sum_{r=0}^{\infty} (1+2r)^{\alpha} \left( \sum_{r_1 + \cdots + r_d = r} \prod_{i=1}^{d} \frac{(2r_i)!}{(r_i)!^{2^{2r_i}}} \right) \left( \int_{0}^{1} Q_{H}(z)^{m} \frac{dz}{z^dH} \right)^{N},$$

because $H_{2n}(0) = \frac{1}{(2\pi)^d} H_{2n+1}(0) = 0$.

By the continuity of the norm, it is not necessary to prove the convergence of this series.

**Remark 9** Xiao and Zhang proved that when $Hd < 1$, that is $\lambda < 0$, $B^{H}$ has a jointly continuous local time.
5  Renormalization of the local time when the time tends to infinity

We can also deduce the behavior of the local time \( L(t, x) \) when \( t = t_1 \cdots t_N \to \infty \) and \( |x| \to 0 \). We also have to distinguish the three cases, namely \( \lambda > 0, \lambda = 0 \) and \( \lambda < 0 \).

The precise result is the following:

**Theorem 10** Let \( \{L(t, x) : t \in [0, \infty)^N, x \in \mathbb{R}^d \} \) be the local time of the \((N, d)-\text{fBm} B^H\). Let \( \lambda = d - \frac{1}{H} \). Then the following limits hold for any \( \alpha < \frac{N}{2H} - \frac{d}{2} \):

1) For \( \lambda > 0 \),

\[
\lim_{t \to \infty, |x| \to 0} \|L(t, x)\|_{\alpha, 2} \left( \frac{2^\frac{\lambda}{2}(\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)! (\log \frac{2t^{2H}}{|x|^2})^{N-1} \Gamma\left(\frac{\lambda}{2}\right)} \right)^{-1} = 1.
\]

2) For \( \lambda = 0 \),

\[
\lim_{t \to \infty, |x| \to 0} \|L(t, x)\|_{\alpha, 2} \left( \frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N! (\log \frac{2t^{2H}}{|x|^2})^{N}} \right)^{-1} = 1.
\]

3) For \( \lambda < 0 \),

\[
\lim_{t \to \infty, |x| \to 0} \|L(t, x)\|_{\alpha, 2} \left( t^{1-dH} \|L(\tilde{1}, 0)\|_{\alpha, 2} \right)^{-1} = 1.
\]

**Proof:** From the scaling property of the \((N, d)-\text{fBm} \) with all the elements \( H_{i,j} \) of the matrix of Hurst parameters equals to \( H \), one can show that the two processes

\[
\{L(t, x) : t \in [0, \infty)^N, x \in \mathbb{R}^d \}
\]

and

\[
\left\{\prod_{j=1}^{N} t_j^{1-dH} L(\tilde{1}, (t_1 \ldots t_N)^{-H} x) : t \in [0, \infty)^N, x \in \mathbb{R}^d \right\}
\]

have the same law.

Hence we have

\[
\|L(t, x)\|_{\alpha, 2}^2 = t^{2(1-dH)} \|L(\tilde{1}, \tilde{t}^{-H} x)\|_{\alpha, 2}^2.
\]

and the conclusion follows from the results of the previous section. \( \square \)

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