Renormalization of the local time for the d-dimensional fractional Brownian motion with N parameters.

M. Eddahbi[1]
Département de Mathématiques et Informatique
Faculté des Sciences et Techniques, Université
Cadi Ayyad, B.P. 549 Marrakech, Maroc

R. Lacayo, J.L. Solé[2] and J. Vives[2]
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain

C.A. Tudor
Samos/Matisse. 90, rue de Tolbiac
Université de Panthéon-Sorbonne. Paris 1
75634 Paris Cedex 13, France

Abstract

We study the asymptotic behavior in Sobolev norm of the local time of the d–dimensional fractional Brownian motion with N–parameters when the space variable tends to zero, both for the fixed time case and when simultaneously time tends to infinity and space variable to zero.

AMS 1991 subject classifications: 60H05, 60H10

1 Introduction

Let $B^H = \{B_t^H : t \geq 0\}$ be a standard fractional Brownian motion (fBm for brevity) with Hurst parameter $H \in (0, 1)$. It is well known that this process is a centered Gaussian process which admits an integral representation of the form

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where W is an standard Wiener process. The kernel $K_H(t, s)$ is given, for $s < t$, by

$$K_H(t, s) = c_H(t - s)^{\mu} - \mu c_H \int_s^t (r - s)^{\mu - 1}(1 - (\frac{s}{r})^{-\mu})dr, \qquad (1)$$

with c_H being a constant and $\mu = H - \frac{1}{2}$.

The covariance function of B_t^H can be represented as

[1]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant SB2000–0060.
[2]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant BFM2000–0009 and by the Catalan CIRIT: Grant 2001SGR00174.

$$R_H(s,t) = \mathbb{E}(B_s^H B_t^H) = \int_0^{s\wedge t} K_H(t,r)K_H(s,r)dr,$$

and has the explicit form

$$R_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

A very good survey about the fBm is the paper of Nualart [5].

For $\overline{H} = (H_1, \ldots, H_N)$ the $(N,1)$–fBm is defined as

$$B_t^{\overline{H}} = \int_{[0,t]} K_{\overline{H}}(t,s)dW_s,$$

where $K_{\overline{H}}(t,s) = \bigotimes_{j=1}^N K_{H_j}(t_j,s_j)$, $s,t \in \mathbb{R}_+^N$ and W is an standard N–parameter Brownian motion. Its covariance function is

$$R_{\overline{H}}(s,t) = \mathbb{E}(B_s^{\overline{H}} B_t^{\overline{H}}) = \prod_{j=1}^N R_{H_j}(s_j,t_j).$$

Finally given the $N \times d$–matrix $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ where for $i = 1, \ldots, d$ and $j = 1, \ldots, N$, $\overline{H}_i = (H_{i,1}, \cdots, H_{i,N})$ is a column vector and $H_{i,j} \in (0,1)$, the N–parameter, d–dimensional fractional Brownian motion ((N,d)–fBm for brevity) is defined by $B^{\overline{H}} = (B_t^{\overline{H}_1}, \ldots, B_t^{\overline{H}_d})_{t\in\mathbb{R}_+^N}$ where its components are independent and for every $i = 1, \ldots, d$, $B^{\overline{H}_i}$ is a $(N,1)$–fBm with Hurst parameter $\overline{H}_i$.

For any $t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d$, the local time $L(t,x)$ of the (N,d)–fBm can be defined as the density of the occupation measure μ_t, defined as

$$\mu_t(A) = \int_{[0,t]} \mathbb{1}_A(B_s^{\overline{H}})ds, \ A \in \mathcal{B}(\mathbb{R}^d).$$

Formally, we can write

$$L(t,x) = \int_{[0,t]} \delta_x(B_s^{\overline{H}})ds$$

where δ_x denotes the Dirac function and $\delta_x(B_s^{\overline{H}})$ is therefore a distribution in the Watanabe sense (see [6]).

This local time for (N,d)–fBm has been studied by Xiao and Zhang [7], Hu and Oksendal [2] and Eddahbi et al. [1] between others.

The aim of this paper is to study the asymptotic behavior of $L(t,x)$ when $|x|$, the euclidean norm of x in $\mathbb{R}^d$, goes to 0, both for a fixed time and when the time goes to infinity, and we renormalize his Sobolev norm. We generalize the results of [3] from the (N,d)–standard Brownian motion to the (N,d)–fractional Brownian motion. In the standard Brownian motion case, the covariance function is simply $R_{\frac{1}{2}}(s,t) = s \wedge t$. Here, the control of the covariance function $R_H(s,t)$ for $H \neq \frac{1}{2}$ is the main difficulty.

Section 2 is devoted to the presentation of the problem. In particular we review from [1] the chaotic decomposition of the local time $L(t,x)$ as a functional of the

(N,d)–fBm and its regularity in terms of Sobolev–Watanabe norms. In section 3 we present a list of auxiliary lemmas. Section 4 is devoted to the presentation and proof of the main result, namely the asymptotic behavior of this local time, for fixed t, in the case $H_{i,j}=H$, $\forall\, i,j$, when $|x|$ goes to 0. In section 5 we extend the result to the case $\underline{t}:=t_1\cdots t_N$ going to infinity.

2 Preliminaries and statement of the problem

If F is a square integrable Brownian random variable, it can be represented by its Wiener chaos expansion

$$F=\sum_{n=0}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ denotes the multiple Itô stochastic integral of the symmetric kernel $f_n\in L^2(\mathbb{R}_+^n)$ with respect to the Wiener process W.

If $\mathbf{L}$ is the Ornstein–Uhlenbeck operator

$$\mathbf{L}F=-\sum_{n=0}^{\infty} nI_n(f_n),$$

$p\in(1,\infty)$ and $\alpha\in\mathbb{R}$, we define the Sobolev–Watanabe spaces $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p}=\|(\mathbf{Id}-\mathbf{L})^{\frac{\alpha}{2}}F\|_{L^p(\Omega)},$$

where $\mathbf{Id}$ stands for the identity mapping.

We denote by D the chaotic derivative operator. It acts on multiple Itô stochastic integrals as

$$D_t(I_n(f_n))=nI_{n-1}(f_n(\cdot,t)),$$

and is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(L^2(\mathbb{R}_+))$.

It is known that a Brownian random variable F belongs to $\mathbb{D}^{\alpha,2}$ if and only if its chaotic decomposition $\sum_{n=0}^{\infty} I_n(f_n)$ satisfies

$$\sum_{n=0}^{\infty}(1+n)^{\alpha}||I_n(f_n)||_2^2<\infty,$$

where $||I_n(f_n)||_2^2=n!||f_n||_2^2$.

Set $\mathbb{D}^{\infty,2}=\cap_{\alpha\in\mathbb{R}}\mathbb{D}^{\alpha,2}$. If $F\in\mathbb{D}^{\infty,2}$, we can compute its chaos expansion using the Stroock formula

$$F=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(\mathbb{E}(D^nF)).$$

For a complete survey of this subjects we refer the reader to the book of Watanabe [6].

Let $p_\varepsilon(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. Consider also, for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the Gaussian kernel on $\mathbb{R}^d$ given by

$$p^d_\varepsilon(x) = \prod_{i=1}^d p_\varepsilon(x_i),\ x = (x_1, \ldots, x_d).$$

We denote by $\mathbf{H}_n$ the n–th Hermite polynomial, defined for $n \geq 1$, by

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp(\frac{x^2}{2}) \frac{d^n}{dx^n}(\exp(-\frac{x^2}{2})),\ x \in \mathbb{R}$$

and $\mathbf{H}_0(x) = 1$.

As we proved in [1] the chaotic decomposition of the local time of the (N, d)–fBm is

$$L(t, x) = \sum_{n_1,\ldots,n_d \geq 0} \int_{[0,t]} \prod_{i=1}^d \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i}} \mathbf{H}_{n_i}(\frac{x_i}{\underline{s}^{\overline{H}_i}}) I^i_{n_i}(K_{\overline{H}_i}(s, \cdot)^{\otimes n_i})\, ds,$$

provided that $\sum_{j=1}^N \frac{1}{H^*_j} > d$, where $t \in \mathbb{R}^N_+$, $x \in \mathbb{R}^d$, $\underline{s} = s_1 \cdots s_N$ and $\underline{s}^{\bar{H}_i} = \prod_{j=1}^N s_j^{H_{i,j}}$. The integrals $I^i_{n_i}$ denotes the multiple Itô stochastic integrals with respect to the independent N–parameter Wiener processes W^i. Finally $H^*_j = \max\{H_{i,j}, i = 1, \ldots, d\}$.

Moreover, in [1] we proved that this functional belongs to the space $\mathbb{D}^{\alpha,2}$ if

$$\alpha < \sum_{j=1}^N \frac{1}{2H^*_j} - \frac{d}{2}.$$

If all $H_{i,j} = H$, this expression becomes $\alpha < \frac{N}{2H} - \frac{d}{2}$, and then a sufficient condition for the local time to be in $L^2(\Omega)$ is $N > Hd$. Observe that this sufficient condition is also founded in Xiao and Zhang [7]. From now on we will suppose always this condition.

Recall that if $H = \frac{1}{2}$,

$$\sum_{j=1}^N \frac{1}{2H^*_j} - \frac{d}{2} = N - \frac{d}{2},$$

which is the same condition obtained in [3] for the N–parameter Wiener process in $\mathbb{R}^d$.

3 Auxiliary lemmas

Lemma 1 *If $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ we have*

$$\sup_{x\in\mathbb{R}} |\sqrt{n!}\mathbf{H}_n(x) e^{-\beta x^2}| \leq c(n \vee 1)^{-\frac{8\beta-1}{12}}$$

Proof: This result is proved in [4]. □

Remark 2 *The factor $\sqrt{n!}$ appears because we do not use the same definition of Hermite polynomials as in [4].*

Lemma 3 *Let $d \geq 1$ and $\nu \in (0,1)$. We can choose a universal constant c such that for any $m \geq 1$,*

$$\sum_{n_1+\cdots+n_d=m} \prod_{i=1}^{d} (n_i \vee 1)^{-\nu} \leq c m^{d(1-\nu)-1}$$

Proof: This result is proved in [4]. □.

Lemma 4 *Let γ and a be positive constants and $b \in \mathbb{R}$. Set $\alpha = \frac{b-1}{a}$. Then*

$$\int_{[0,1]^N} \exp(-\frac{\gamma}{\underline{s}^a}) \frac{ds}{\underline{s}^b} = \frac{1}{(N-1)!} (\frac{1}{a})^N \gamma^{-\alpha} g_{N-1}(\gamma, \alpha)$$

where

$$g_{N-1}(\gamma,\alpha) := \int_{\gamma}^{\infty} t^{\alpha-1} e^{-t} (\log \frac{t}{\gamma})^{N-1} dt.$$

Proof:

Using the change of variables $u_1 = s_1 \cdots s_N, u_2 = s_2 \cdots s_N, \ldots, u_N = s_N$, with Jacobi determinant $\frac{1}{u_2 \cdots u_N}$, we obtain

$$\begin{aligned}\int_{[0,1]^N} \exp(-\frac{\gamma}{\underline{s}^a}) \frac{ds}{\underline{s}^b} &= \int_{\{0 \leq u_1 \leq \cdots \leq u_N \leq 1\}} \frac{1}{u_1^b} \exp(-\frac{\gamma}{u_1^a}) \frac{du_N \cdots du_2}{u_N \cdots u_2} du_1 \\ &= \frac{1}{(N-1)!} \int_0^1 (\log \frac{1}{r})^{N-1} \frac{1}{r^b} \exp(-\frac{\gamma}{r^a}) dr,\end{aligned}$$

and making the change of variable $\gamma r^{-a} = t$ we get the desired result. □

Lemma 5 *The function*

$$Q_H(z) = \begin{cases} \dfrac{R_H(1,z)}{z^H} & \text{if } z \in (0,1] \\ 0 & \text{if } z = 0, \end{cases}$$

has the following properties:

1. *It is strictly increasing and it continously maps $[0,1]$ onto $[0,1]$. Moreover, $Q_H(1) = 1$.*
2. *For fixed $\delta \in (0,1)$ and for any $z \in [0, 1-\delta]$, it satisfies the inequality*

$$Q_H(z) \leq c(H,\delta) z^G,$$

where $G = H \wedge (1-H)$.

3. For fixed $\delta \in (0,1)$ *and* $\beta > 0$, *it satisfies the inequality*

$$\int_{1-\delta}^{1} Q_H(z)^{\beta} dz \leq \frac{c(H,\delta)}{\beta^{\frac{1}{2H}}}.$$

Proof:

The proof of parts 1 and 3 are done in [1].

For the part 2 we have

$$Q_H(z) = \frac{1-(1-z)^{2H}}{2z^H} + \frac{z^H}{2}.$$

Using Taylor expansion, $1-(1-z)^{2H} = 2H(1-\theta)^{2H-1}z$ with $0 \leq \theta \leq z$.

If $H \geq \frac{1}{2}$, we have $1-(1-z)^{2H} \leq 2Hz$, and therefore $Q_H(z) \leq Hz^{1-H} + \frac{1}{2}z^H \leq c_1 z^{1-H}$, c_1 being a positive constant.

If $H < \frac{1}{2}$ and $z \in [0, 1-\delta]$, we have $1-(1-z)^{2H} \leq 2H\delta^{2H-1}z$, and then $Q_H(z) \leq H\delta^{2H-1}z^{1-H} + \frac{1}{2}z^H \leq c_2 z^H$, c_2 being another positive constant. □

In what follows, for every $x > 0$ and $\gamma \geq 0$, we denote the complementary incomplete Gamma function as

$$\Gamma(x,\gamma) = \int_{\gamma}^{\infty} e^{-t} t^{x-1} dt.$$

In particular $\Gamma(x) := \Gamma(x,0)$ and $\Gamma(x,\gamma) \leq \Gamma(x)$.

Lemma 6 *The function*

$$g_{N-1}(\gamma,\alpha) := \int_{\gamma}^{\infty} t^{\alpha-1} e^{-t} (\log \frac{t}{\gamma})^{N-1} dt.$$

has the following behavior when γ *tends to* 0 :

1. *If* $\alpha > 0, g_{N-1}(\gamma,\alpha) = (\log \frac{1}{\gamma})^{N-1}\Gamma(\gamma,\alpha) + \mathcal{O}((\log \frac{1}{\gamma})^{N-2})$.

2. *If* $\alpha = 0, g_{N-1}(\gamma,\alpha) = e^{-\gamma}\frac{1}{N}(\log \frac{1}{\gamma})^{N} + \mathcal{O}((\log \frac{1}{\gamma})^{N-1})$.

3. *If* $\alpha < 0, g_{N-1}(\gamma,\alpha) = \gamma^{\alpha}(\frac{\Gamma(N)}{|\alpha|^N} + o(\gamma))$.

Proof:

Note first that

$$(\log \frac{t}{\gamma})^{N-1} = \sum_{k=0}^{N-1} \binom{N-1}{k} (\log \frac{1}{\gamma})^{N-1-k} (\log t)^k. \qquad (2)$$

Then,

- If $\alpha > 0$, the function

$$t \longmapsto t^{\frac{\alpha}{2}-1}e^{-t}(\log t)^k$$

is always integrable on $[0,\infty)$ for any $k \in \mathbb{N}$. Therefore,

$$g_{N-1}(\gamma,\alpha) = (\log\frac{1}{\gamma})^{N-1}\Gamma(\gamma,\alpha) + \mathcal{O}((\log\frac{1}{\gamma})^{N-2}).$$

- If $\alpha = 0$, we need to estimate the integral

$$g_{N-1}(\gamma,0) = \int_\gamma^\infty t^{-1}e^{-t}(\log\frac{t}{\gamma})^{N-1}dt.$$

Integrating by parts we obtain

$$g_{N-1}(\gamma,0) = \frac{1}{N}\int_\gamma^\infty e^{-t}(\log\frac{t}{\gamma})^N dt = \frac{e^{-\gamma}}{N}(\log\frac{1}{\gamma})^N + \mathcal{O}((\log\frac{1}{\gamma})^{N-1})$$

- If $\alpha < 0$, making the change of variable $s = -\alpha\log(\frac{t}{\gamma})$, the result follows immediately.

□

4 Renormalization of the local time for fixed t

The main purpose of this section is to study the asymptotic behavior of $L(t,x)$, for $t \in \mathbb{R}^N$ and $x \in \mathbb{R}^d$, when $|x| \to 0$. In the case $dH \geq 1$ it has a singularity. An interesting question is to renormalize the local time, that means, to find a deterministic function $f(t,x)$ such that $f(t,x)L(t,x)$ converge to 1 in some precise sense. We will do it with respect the norm $\|\cdot\|_{\alpha,2}$. Then we will obtain a function $f(t,x)$ such that $\|f(t,x)L(t,x)\|_{\alpha,2}$ converges to 1 when $|x| \to 0$, both for fixed t and when $\underline{t} = t_1 \cdots t_N \to \infty$.

Recall the expression of the $\mathbb{D}^{\alpha,2}$–norm of the local time $L(t,x)$. For the sake of simplicity we take $t := (1,\ldots,1)$.

We have

$$\|L(\widetilde{1},x)\|^2_{\alpha,2} = \sum_{m=0}^\infty (1+m)^\alpha A_m(x), \tag{3}$$

where

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \|\int_{[0,t]}\prod_{i=1}^d \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i\overline{H}_i}}\mathbf{H}_{n_i}(\frac{x_i}{\underline{s}^{\overline{H}_i}})I^i_{n_i}(K_{\overline{H}_i}(s,\cdot)^{\otimes n_i})\,ds\|^2_{L^2(\Omega)},$$

and as

$$E(I^i_{n_i}(K_{\bar{H}_i}(u,\cdot)^{\otimes n_i})I^j_{n_j}(K_{\bar{H}_j}(v,\cdot)^{\otimes n_j})) = \delta_{ij} n_i! \big(R_{\bar{H}_i}(u,v)\big)^{n_i},$$

$$\begin{aligned} A_m(x) &= \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^{d} (\prod_{j=1}^{N} \frac{R_{H_{i,j}}(u_j,v_j)}{(u_j v_j)^{H_{i,j}}})^{n_i} \\ &\quad \times n_i!\ \mathbf{H}_{n_i}(\frac{x_i}{\underline{u}^{\overline{H}_i}})\mathbf{H}_{n_i}(\frac{x_i}{\underline{v}^{\overline{H}_i}}) p_{\underline{u}^{2\overline{H}_i}}(x_i) p_{\underline{v}^{2\overline{H}_i}}(x_i), \end{aligned}$$

and in particular

$$A_0(x) = (\int_{[0,1]^N} ds \prod_{i=1}^{d} \frac{1}{(2\pi \prod_{j=1}^{N} s_j^{2H_{i,j}})^{\frac{1}{2}}} \exp(-\frac{x_i^2}{2\prod_{j=1}^{N} s_j^{2H_{i,j}}}))^2.$$

In all this section we confine our attention to the situation where $H_{i,j} = H$ for all $(i,j) \in \{1,\ldots,d\} \times \{1,\ldots,N\}$, and use the notation B^H for $B^{\overline{H}}$.

Observe that in this particular case

$$A_0(x) = \frac{1}{(2\pi)^d} \Big(\int_{[0,1]^N} \frac{1}{\underline{s}^{dH}} \exp\big(-\frac{|x|^2}{2\underline{s}^{2H}}\big) ds \Big)^2,$$

and

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} \int_{[0,1]^N} (\prod_{j=1}^{N} \frac{R_H(u_j,v_j)}{(u_j v_j)^H})^m \prod_{i=1}^{d} n_i! \mathbf{H}_{n_i}(\frac{x_i}{\underline{u}^H}) \mathbf{H}_{n_i}(\frac{x_i}{\underline{v}^H}) p_{\underline{u}^{2H}}(x_i) p_{\underline{v}^{2H}}(x_i) du dv.$$

Our main result is the following:

Theorem 7 *Let B^H be (N,d)–fBm. Set $\lambda := d - \frac{1}{H}$. For any $\alpha < \frac{N}{2H} - \frac{d}{2}$ we have:*

1) If $\lambda > 0$, $\lim_{|x|\to 0} \|L(1,x)\|_{\alpha,2} \Big(\frac{2^{\frac{\lambda}{2}} (\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)!} (\log \frac{2}{|x|^2})^{N-1} \Gamma(\frac{\lambda}{2}) \Big)^{-1} = 1.$

2) If $\lambda = 0$, $\lim_{|x|\to 0} \|L(1,x)\|_{\alpha,2} \Big(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!} (\log \frac{2}{|x|^2})^N \Big)^{-1} = 1.$

3) If $\lambda < 0$,

$$\begin{aligned} \lim_{|x|\to 0} \|L(1,x)\|_{\alpha,2} &= \|L(1,0)\|_{\alpha,2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \Big(\frac{1}{1-Hd}\Big)^{\frac{N}{2}} \\ &\times \Big[\sum_{r=0}^{\infty} (1+2r)^{\alpha} \Big(\sum_{r_1+\cdots+r_d=r} \prod_{i=1}^{d} \frac{(2r_i)!}{(r_i)! 2^{2r_i}} \Big(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \Big)^N \Big]^{\frac{1}{2}} < \infty. \end{aligned}$$

Remark 8 *This theorem shows that for $\lambda \geq 0$ the local time explodes at the origin and for $\lambda < 0$ it does not. Observe that if $H = \frac{1}{2}$, we have that the local time explodes at the origin if and only if $d \geq 2$, as is discussed in [3].*

Proof:

The idea of the proof is to show that the convergence of $A_m(x)$ for any $m \geq 1$ when $|x| \to 0$, is controlled by $A_0(x)$ and then the asymptotic behavior of $L(1,x)$ coincides with the asymptotic behavior of $A_0(x)^{\frac{1}{2}}$.

Define now, for $\gamma > 0$ and $m \geq 0$,

$$B_m(\gamma) = \int_{[0,1]^N} \int_{[0,1]^N} \frac{(\prod_{j=1}^N R_H(u_j, v_j))^m}{(\underline{u} \cdot \underline{v})^{H(m+d)}} \exp(-\frac{\gamma}{\underline{u}^{2H}}) \exp(-\frac{\gamma}{\underline{v}^{2H}}) dudv.$$

Clearly,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0(\frac{1}{2}|x|^2).$$

For $m \geq 1$, choosing $\beta \in [\frac{1}{4}, \frac{1}{2})$, we can write

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} \int_{[0,1]^N} \Big(\prod_{j=1}^N \frac{R_H(u_j, v_j)}{(u_j v_j)^H} \Big)^m \frac{1}{(\underline{uv})^{dH}}$$

$$\times \prod_{i=1}^d \sqrt{n_i!}\, \mathbf{H}_{n_i}(\frac{x_i}{\underline{u}^H}) \exp\{-\beta \frac{x_i^2}{\underline{u}^{2H}}\} \sqrt{n_i!}\, \mathbf{H}_{n_i}(\frac{x_i}{\underline{v}^H}) \exp\{-\beta \frac{x_i^2}{\underline{v}^{2H}}\}$$

$$\times \exp\{-(\frac{1}{2} - \beta) \frac{x_i^2}{\underline{u}^{2H}}\} \exp\{-(\frac{1}{2} - \beta) \frac{x_i^2}{\underline{v}^{2H}}\} dudv,$$

and applying Lemmas 1 and 2 we obtain

$$A_m(x) \leq c \frac{1}{(2\pi)^d} m^{d(1-\frac{8\beta-1}{6})-1} B_m((\frac{1}{2} - \beta)|x|^2).$$

Then our problem reduces to the study of the asymptotic behavior of B_m.

As $R_H(u_j, v_j) = R_H(1, \frac{v_j}{u_j}) u_j^{2H}$, we have

$$B_m(\gamma) = 2^N \int_{[0,1]^N} \int_0^{u_N} \cdots \int_0^{u_1} \prod_{j=1}^N \frac{R_H(1, \frac{v_j}{u_j})^m u_j^{2Hm}}{(u_j v_j)^{H(m+d)}} \exp(-\frac{\gamma}{\underline{u}^{2H}}) \exp(-\frac{\gamma}{\underline{v}^{2H}}) dv_N \cdots dv_1.$$

With the change $\frac{v_j}{u_j} = z_j, \forall\, j = 1, \ldots, N$ and computing iteratively the previous integral, we find

$$B_m(\gamma) = 2^N \int_{[0,1]^N} (\int_{[0,1]^N} \underline{u}^{1-2Hd} \exp(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}) du_1 \cdots du_N)) \prod_{j=1}^N \frac{R_H(1, z_j)^m}{{z_j}^{H(m+d)}} dz_1 \cdots dz_N$$

where $\kappa(r) = 1 + \frac{1}{r^{2H}}$.

By Lemma 4, with $a = 2H$ and $b = 2Hd - 1$, we have

$$J_N(\gamma, \underline{z}) = \int_{[0,1]^N} \underline{u}^{1-2Hd} \exp(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}) du = c(N,d,H)\gamma^{-\lambda} \int_\gamma^\infty e^{-s\kappa(\underline{z})} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} ds,$$

where $\lambda = d - \frac{1}{H} = \frac{b-1}{a}$.

Therefore

$$B_m(\gamma) = c(N,H,d)\gamma^{-\lambda} \int_\gamma^\infty \int_{[0,1]^N} \prod_{j=1}^N \frac{R_H(1,z_j)^m}{z_j^{Hm}} \cdot \frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}} e^{-s} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} dz ds.$$

First we will see that for $m > \frac{\lambda H}{G}$, we have

$$B_m(\gamma) \le c(H,d,N)\gamma^{-\lambda} g_{N-1}(\gamma,\lambda) m^{-\frac{N}{2H}}. \tag{4}$$

Indeed, controlling the exponential by 1, we obtain

$$\begin{aligned} B_m(\gamma) &\le c(N,H,d)\gamma^{-\lambda} \int_{[0,1]^N} \prod_{j=1}^N \frac{R_H(1,z_j)^m}{z_j^{H(m+d)}} \int_\gamma^\infty e^{-s} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} dz ds \\ &= c(N,H,d)\gamma^{-\lambda} g_{N-1}(\gamma,\lambda) (\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz)^N, \end{aligned}$$

where the function Q_H is introduced in lemma 5.

Now, choosing $\delta \in (0,1)$, we have

$$\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz \le \int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz + (1-\delta)^{-dH} \int_{1-\delta}^1 Q_H(z)^m dz$$

The second summand on the right, using part 3 of Lemma 4, is controlled by $c(H,\delta) m^{-\frac{1}{2H}}$.

For the first summand, if $m > \frac{dH-1}{G} = \frac{\lambda H}{G}$, we fix $\alpha \in (\frac{\lambda H}{G}, m)$, and write

$$\int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz = \int_0^{1-\delta} Q_H(z)^{m-\alpha} Q_H(z)^\alpha \frac{1}{z^{dH}} dz.$$

Using that Q_H is an increasing function and part 2 of Lemma 5, we control this by

$$Q_H(1-\delta)^{m-\alpha} c(H,\delta,\alpha) \int_0^{1-\delta} z^{\alpha G - dH} dz.$$

As $\alpha > \frac{\lambda H}{G}$, the integral that appears in the last expression is a constant that depends on H, d, α and δ.

Therefore, being $Q_H(1-\delta)<1$, we can estimate this term by

$$c(H,d,\delta,\alpha)m^{-\frac{1}{2H}},$$

and we get (4).

Note that this result is true only for $m>\frac{\lambda H}{G}$. If $\lambda\le 0$ this covers all cases. But if $\lambda>0$ the B_m terms with $m\le\frac{\lambda H}{G}$ are not controlled yet. The following part of the proof will discuss these first terms.

Observe that for any $0<\epsilon<m$, being $Q_H(\cdot)\le 1$, we have

$$B_m(\gamma)\le B_\epsilon(\gamma)$$

Now we will see that for $\lambda>0$,

$$B_\epsilon(\gamma)\le c(H,d,N)\gamma^{-\lambda}g_{N-1}(\gamma,\alpha)$$

where α is some positive constant depending also on ϵ.

Indeed, putting $c=c(N,H,d)$,

$$B_\epsilon(\gamma)=c\gamma^{-\lambda}\int_\gamma^\infty\int_{[0,1]^N}\prod_{j=1}^N Q_H(z_j)^\epsilon\cdot\frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}}e^{-s}s^{\lambda-1}(\log\frac{s}{\gamma})^{N-1}dzds$$

$$=c\gamma^{-\lambda}\int_\gamma^\infty\sum_{k=0}^N\binom{N}{k}\underbrace{\int_0^{1-\delta}\cdots\int_0^{1-\delta}}_{k}\underbrace{\int_{1-\delta}^1\cdots\int_{1-\delta}^1}_{N-k}\prod_{j=1}^N Q_H(z_j)^\epsilon\cdot\frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}}e^{-s}s^{\lambda-1}(\log\frac{s}{\gamma})^{N-1}dzds,$$

because the function

$$\prod_{j=1}^N Q_H(z_j)^\epsilon\cdot\frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}},$$

is symmetric in z.

Now estimating Q_H and the exponential by 1 in the integrals between $1-\delta$ and 1, we obtain

$$B_\epsilon(\gamma)\le c\gamma^{-\lambda}\int_\gamma^\infty\sum_{k=0}^N\binom{N}{k}\int_0^{1-\delta}\cdots\int_0^{1-\delta}\frac{\delta^{N-k}}{(1-\delta)^{dH(N-k)}}$$

$$\times\prod_{j=1}^k Q_H(z_j)^\epsilon\cdot\frac{e^{-\frac{s}{(z_1\cdots z_k)^{2H}}}}{(z_1\cdots z_k)^{Hd}}e^{-s}s^{\lambda-1}(\log\frac{s}{\gamma})^{N-1}dz_1\cdots dz_k)ds$$

$$\le c\gamma^{-\lambda}\int_\gamma^\infty\sum_{k=0}^N\binom{N}{k}\int_0^{1-\delta}\cdots\int_0^{1-\delta}(\frac{\delta}{(1-\delta)^{dH}})^{N-k}c(H,\delta)^{k\epsilon}$$

$$\times \prod_{j=1}^{k} z_j^{\epsilon G - dH} e^{-\frac{s}{(z_1\cdots z_k)^{2H}}} e^{-s} s^{\lambda-1} (\log \frac{s}{\gamma})^{N-1} dz_1 \dots dz_k ds,$$

where we have used section 2 of Lemma 5.

Now, choosing $\epsilon < \frac{dH}{G}$, we can use Lemma 4 with $a = 2H, b = -\epsilon G + dH, \gamma = s, N = k$ and $\alpha = \frac{dH - \epsilon G - 1}{2H}$, to bound the right hand side of the last inequality by

$$c\gamma^{-\lambda} \int_\gamma^\infty \sum_{k=0}^{N} \int_s^\infty (\log \frac{t}{s})^{k-1} s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1} t^{\frac{\lambda}{2} - \frac{\epsilon G}{2H} - 1} e^{-t} e^{-s} (\log \frac{s}{\gamma})^{N-1} dt ds,$$

where c is a constant that depends on $H, d, N, \epsilon, k, \delta$.

Using the fact that for any $n \geq 1$ and for $t \geq s$ we have

$$\log \frac{t}{s} \leq n \Big(\frac{t}{s}\Big)^{\frac{1}{n}},$$

and taking $n = M(k-1)$ for a big M, we obtain

$$B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \sum_{k=0}^{N} \int_\gamma^\infty \int_s^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} t^{\frac{\lambda}{2} - \frac{\epsilon G}{2H} - 1 + \frac{1}{M}} e^{-t} e^{-s} (\log \frac{s}{\gamma})^{N-1} dt ds,$$

where c depends also on M. From now on in each expression c will be the suitable constant.

As $\epsilon < m < \frac{\lambda H}{G}$, we have $\frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M} > 0$ and

$$B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \sum_{k=0}^{N} c \int_\gamma^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} \Gamma(s, \frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M}) (\log \frac{s}{\gamma})^{N-1} ds.$$

Controlling the truncated Gamma function by the corresponding Gamma function we obtain

$$\begin{aligned} B_\epsilon(\gamma) &\leq c\gamma^{-\lambda} \int_\gamma^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} (\log \frac{s}{\gamma})^{N-1} ds \\ &= c\gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}). \end{aligned}$$

Observe that for M sufficiently large

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

Finally for the $m = 0$ case, using Lemma 4, we have immediately, as $\alpha = \frac{\lambda}{2}$

$$B_0(\gamma) = \frac{1}{((N-1)!)^2} \frac{1}{(2H)^{2N}} \gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2})^2.$$

Therefore we have to separate the cases $\lambda \geq 0$ and $\lambda < 0$.
For $\lambda \geq 0$ we have

$$\|L(\widetilde{1}, x)\|_{\alpha,2}^2 = \sum_{m=0}^{\infty} (1+m)^{\alpha} A_m(x)$$

- The terms A_m with $m = 1, \ldots, [\frac{\lambda H}{G}]$ are controlled by

$$c\gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}) m^{d(1-\frac{8\beta-1}{6})-1}$$

  where $\gamma = (\frac{1}{2} - \beta)|x|^2$, and ε and M satisfy

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

  Then, by Lemma 6, part 1, this is asymptotically, when $\gamma \downarrow 0$, as

$$c\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{N-1}$$

- The terms A_m with $m \geq [\frac{\lambda H}{G}] + 1$ are controlled by

$$cm^{d(1-\frac{8\beta-1}{6})-1}\gamma^{-\lambda} g_{N-1}(\gamma, \lambda) m^{-\frac{N}{2H}}.$$

  Then

$$\sum_{m>\frac{\lambda H}{d}} (1+m)^{\alpha} A_m(x) \leq c\big[\sum_{m>\frac{\lambda H}{d}} m^{d(1-\frac{8\beta-1}{6})-1} m^{-\frac{N}{2H}} (1+m)^{\alpha}\big]\gamma^{-\lambda} g_{N-1}(\gamma, \lambda),$$

  and using the fact that $\alpha < \frac{N}{2H} - \frac{d}{2}$, we have that the series between keys is convergent and the asymptotic behavior of the last expression is, by lemma 6, as

$$c\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{N-1}.$$

  Finally,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0(\frac{1}{2}|x|^2) = \frac{1}{(2\pi)^d} \frac{1}{((N-1)!)^2} \frac{1}{(2H)^{2N}} \gamma^{-\lambda} g_{N-1}(\gamma, \frac{\lambda}{2})^2,$$

  where $\gamma = \frac{|x|^2}{2}$. When $\gamma \downarrow 0$, this goes to ∞ as $\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{2N-2}$, and as the exponent of the logarithm is 2N-2, this term dominates the asymptotical behavior. Note that we consider $A_0^{\frac{1}{2}}$ in place of A_0, to get the functions that appear in the theorem.

The $\lambda < 0$ case follows directly. As we have seen before,

$$\sum_{m\geq 1}(1+m)^{\alpha}A_m(x)$$

is controlled by $\gamma^{-\lambda}g_{N-1}(\gamma,\lambda)$, and by Lemma 6, part 3, this term goes to a constant when $\gamma \downarrow 0$.

In this case the norm $\|L(t,x)\|_{\alpha,2}$ is continous. Therefore we don't have an explosion, and

$$\lim_{|x|\to 0}\|L(1,x)\|_{\alpha,2} = \|L(1,0)\|_{\alpha,2} = \big(\sum_{m=0}^{\infty}(1+m)^{\alpha}A_m(0)\big)^{\frac{1}{2}},$$

where

$$A_m(0) = \frac{1}{(2\pi)^d}\Big(\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d} n_i!\mathbf{H}_{n_i}(0)^2\Big)B_m(0),$$

and

$$\begin{aligned} B_m(0) &= 2^N\int_{[0,1]^N}\Big(\int_{[0,1]^N}\underline{u}^{1-2Hd}du_1\cdots du_N)\Big)\prod_{j=1}^{N}\frac{R_H(1,z_j)^m}{z_j^{H(m+d)}}dz_1\cdots dz_N. \\ &= 2^N\Big(\int_0^1 u^{1-2Hd}du\Big)^N\Big(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\Big)^N \\ &= \Big(\frac{1}{1-Hd}\Big)^N\Big(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\Big)^N \end{aligned}$$

Note that as $\lambda < 0$, $1-2Hd > -1$.

Finally,

$$\|L(\widetilde{1},0)\|^2_{\alpha,2} = \frac{1}{(2\pi)^d}\Big(\frac{1}{1-Hd}\Big)^N\sum_{m=0}^{\infty}(1+m)^{\alpha}\Big(\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d}n_i!\cdot\mathbf{H}_{n_i}(0)^2\Big)\Big(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\Big)^N$$

$$= \frac{1}{(2\pi)^d}\Big(\frac{1}{1-Hd}\Big)^N\sum_{r=0}^{\infty}(1+2r)^{\alpha}\Big(\sum_{r_1+\cdots+r_d=r}\prod_{i=1}^{d}\frac{((2r_i)!)}{(r_i!)^2 2^{2r_i}}\Big)\Big(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\Big)^N,$$

because $H_{2n}(0) = \frac{1}{2^{2n}(n!)^2}$ and $H_{2n+1}(0) = 0$.

By the continuity of the norm, it is not necessary to prove the convergence of this series.

Remark 9 *Xiao and Zhang proved that when $Hd < 1$, that is $\lambda < 0$, B^H has a jointly continuous local time.*

5 Renormalization of the local time when the time tends to infinity

We can also deduce the behavior of the local time $L(t, x)$ when $\underline{t} = t_1 \cdots t_N \to \infty$ and $|x| \to 0$. We also have to distinguish the three cases, namely $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

The precise result is the following:

Theorem 10 *Let* $\{L(t, x) : t \in [0, \infty)^N, x \in \mathbb{R}^d\}$ *be the local time of the* (N, d)*–fBm* B^H*. Let* $\lambda = d - \frac{1}{H}$*. Then the following limits hold for any* $\alpha < \frac{N}{2H} - \frac{d}{2}$ *:*

1) For $\lambda > 0$*,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \Big(\frac{2^{\frac{\lambda}{2}} (\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)!} (\log \frac{2\underline{t}^{2H}}{|x|^2})^{N-1} \Gamma(\frac{\lambda}{2}) \Big)^{-1} = 1.$$

2) For $\lambda = 0$*,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \Big(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!} (\log \frac{2\underline{t}^{2H}}{|x|^2})^N \Big)^{-1} = 1.$$

3) For $\lambda < 0$*,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \big(\underline{t}^{(1-dH)} \|L(\widetilde{1}, 0)\|_{\alpha,2} \big)^{-1} = 1.$$

Proof: From the scaling property of the (N, d)–fBm with all the elements $H_{i,j}$ of the matrix of Hurst parameters equals to H, one can show that the two processes

$$\{L(t,x) : t \in [0,\infty)^N,\ x \in \mathbb{R}^d\}$$

and

$$\{\prod_{j=1}^N t_j^{1-dH} L(\widetilde{1}, (t_1 \ldots t_N)^{-H} x) : t \in [0,\infty)^N,\ x \in \mathbb{R}^d\}$$

have the same law.

Hence we have

$$\|L(t,x)\|_{\alpha,2}^2 = \underline{t}^{2(1-dH)} \|L(\widetilde{1}, \underline{t}^{-H} x)\|_{\alpha,2}^2.$$

and the conclusion follows from the results of the previous section. □

Acknowledgment. M. Eddahbi is grateful to the *Centre de Recerca Matemàtica (IEC) and the Department of Mathematics of the UAB, Bellaterra, Barcelona*, Spain for the kind support and hospitality he received while this work was carried out.

References

[1] M. Eddahbi, R. Lacayo, J.L. Solé, C. A. Tudor, J. Vives: Regularity of the local time for the d–dimensional fractional Brownian motion with N–parameters. *To appear in Stoc. Analysis and Appl.* (2005)

[2] Y. Hu, B. Oksendal, Chaos expansion of local time of fractional brownian motion. *Stochastic Analysis and Applications* **20, 6** (2002) 815–837

[3] P. Imkeller and F. Weisz, The asymptotic behaviour of local time and occupation integrals of the N–parameter Wiener process in $\mathbb{R}^d$. *Prob. Theo. Rela. Fields.* **98** (1994) 47–75.

[4] P. Imkeller, V. Perez–Abreu and J. Vives, Chaos expansion of double intersection local time of Brownian motion in $\mathbb{R}^d$ and renormalization. *Stoc. Proc. Appl.* **56, 1** (1995) 1–34.

[5] D. Nualart, Stochastic calculus with respect to fractional brownian motion and applications. *Contemporary Mathematics* **336** (2003) 3–39

[6] S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin calculus.* Springer. (1984)

[7] Xiao, Y., Zhang, T., Local times of fractional brownian sheets. *Prob. Theo. Rela. Fields* **124** (2002) 204–226