Itô formula for the two-parameter fractional Brownian motion using the extended divergence operator

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Abstract

We develop a stochastic calculus of divergence type with respect to the fractional Brownian sheet (fBs) with any Hurst parameters in $(0, 1)$ and beyond the fractional scale. We define stochastic integration in the extended Skorohod sense, and derive Itô and Tanaka formulas. In the case of Gaussian fields that are more irregular than fBs for any Hurst parameters, we are able to complete the same program for those Gaussian fields that are almost-surely uniformly continuous.

Key words and phrases: fractional Brownian motion, Brownian sheet, Malliavin calculus, Skorohod integral, Hurst parameter, Gaussian regularity.

1 Introduction

In recent years stochastic integration with respect to Gaussian processes in general and to the fractional Brownian motion (fBm for short) in particular has been intensively studied. Different approaches have been considered in order to develop a stochastic calculus for fBm, including Skorohod (divergence) integration and white noise calculus, pathwise stochastic calculus or the rough path analysis. The most complicated situation is when the fBm's so-called *Hurst* parameter H is small. H is a self-similarity parameter, and is related to the regularity of the fBm. For example, in the Malliavin calculus approach, the integral of the fBm B^H with respect to itself exists if and only if $H > 1/2$. In the pathwise approach, the barrier for the standard definition of the integral is $H = 1/6$. Therefore, for small parameter, one need an extended, relaxed way to integrate. See e.g. [2] or [5]. In the two-parameter case, the Itô formula for the fractional Brownian sheet has been proved in [10] for $H > 1/2$.

This paper is devoted to developing the stochastic integration for the fractional sheet with Hurst parameters less than 1/2, and beyond the fractional scale. We introduce and use a notion of extended divergence that generalize a definition introduced in [2], and in [7] in a wider context. Our techniques and notation are closest to those of [7], but nevertheless in our case we need to pay a particular attention to the Skorohod integrability of the integrands appearing in the Itô formula.

The paper is organized as follows. Section 2 contains preliminaries on the standard and extended Malliavin calculus with respect to the fractional Brownian sheet, including a proof of existence of extended Skorohod integrals. In Section 3 we derive an Itô's formula for $H < 1/2$ and briefly discuss on the local time. Section 4 describes the extension of our calculus beyond the fractional scale, generalizing the approach of [7] to two parameters.

2 Preliminaries

2.1 Malliavin calculus and Wiener integral

Let $T = [0,1]^2$ and let $\left(W^{\alpha,\beta}_{s,t}\right)_{(s,t)\in T}$ be a fractional Brownian sheet with Hurst parameters $\alpha, \beta \in (0,1)$. This process is defined as a centered Gaussian process under some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, starting from zero, and with the covariance function

$$E\left(W^{\alpha,\beta}_{s,t} W^{\alpha,\beta}_{u,v}\right)$$
$$= R^{\alpha,\beta}(s,t,u,v) := \frac{1}{2}\left(s^{2\alpha} + u^{2\alpha} - |s-u|^{2\alpha}\right)\frac{1}{2}\left(t^{2\beta} + u^{2\beta} - |t-v|^{2\beta}\right).$$

Let us briefly recall the framework of the Malliavin calculus for the fractional Brownian sheet. Denote by $\mathcal{H}^{(2)}$ the canonical Hilbert space of $W^{\alpha,\beta}$. That is, $\mathcal{H}^{(2)}$ is the closure of the linear space of the indicator functions $1_{[0,s]\times[0,t]}$, $s,t \in [0,1]$ with respect to the scalar product

$$\langle 1_{[0,s]\times[0,t]}, 1_{[0,u]\times[0,v]}\rangle_{\mathcal{H}^{(2)}} = R^{\alpha,\beta}(s,t,u,v).$$

Let $\mathcal{S}_{\mathcal{H}^{(2)}}$ be the class of 'smooth' random variables of the form

$$F = f\left(W^{\alpha,\beta}(\varphi_1), \ldots, W^{\alpha,\beta}(\varphi_n)\right) \qquad \varphi_i \in \mathcal{H}^{(2)}$$

where f and all its derivatives are bounded. The Malliavin derivative operator acts on random variables F as above in the following way

$$D_{s,t}F = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\left(W^{\alpha,\beta}(\varphi_1), \ldots, W^{\alpha,\beta}(\varphi_n)\right)\varphi_i(s,t) \qquad (s,t) \in T.$$

The operator D is closable and it can be extended to the closure of $\mathcal{S}_{\mathcal{H}^{(2)}}$ with respect to the norm

$$\|F\|_{1,2} = E\|F\|^2_{L^2(\Omega)} + E\|DF\|^2_{L^2(\Omega;\mathcal{H}^{(2)})}.$$

The Skorohod integral is the adjoint of D. Its domain $Dom(\delta)$ is the class of square integrable processes U such that for some constant $C > 0$,

$$|E\langle DF, U\rangle_{\mathcal{H}^{(2)}}| \leq C\|F\|_{L^2(\Omega)} \qquad \forall F \in \mathcal{S}_{\mathcal{H}^{(2)}}.$$

One of the long-standing difficulties with the Skorohod integral is that its domain is sometimes too small. For example, in the one-dimensional case, the fractional Brownian motion B^H is integrable with respect to itself if and only if $H > 1/4$ (see [2]). The same happens in the case of the sheet: the argument of [2] can be used to show that $W^{\alpha,\beta} \in Dom(\delta)$ if and only if both $\alpha > 1/4$ and $\beta > 1/4$. Therefore, an extended divergence is needed for the stochastic integration with respect to the fractional Brownian sheet with small parameters. We use the method of [2] and [7]. It is traditional to introduce some elements of fractional calculus to deal with fBm. Let f be a function on $[0,1]$ and $\alpha > 0$. Then

$$I^\alpha_{b-}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(s)}{(s-t)^{1-\alpha}}ds$$

is the right-sided Riemann-Liouville fractional integral of order α, while the right-sided integral I^α_{a+} is defined using integration from a to t. For $\alpha \in (0,1)$

$$D^\alpha_{b-}f\,(t) := \frac{1}{\Gamma(1-\alpha)}\frac{\delta}{\delta t}\int_t^b \frac{f(s)}{(s-t)^{\alpha}}ds.$$

is the left-sided Riemann–Liouville fractional derivative of order α; I^α_{b-} and D^α_{b-} are inverses of each other. We introduce the operator

$$\left[K^\star_{\beta,t}f\right](s) := c_\beta s^{\frac{1}{2}-\beta}\left(I^{\beta-\frac{1}{2}}_{t-}\left[(\cdot)^{\beta-\frac{1}{2}}f(\cdot)\right]\right)(s) \tag{1}$$

where

$$c_\beta = \sqrt{\frac{2\beta(\beta-\frac{1}{2})\Gamma(\beta-\frac{1}{2})^2}{\mathrm{B}(\beta-\frac{1}{2}, 2-2\beta)}}.$$

We set

$$K^{\star,2}_{\alpha,\beta,t,s} := K^\star_{\alpha,t}\otimes K^\star_{\beta,s}.$$

In the sequel we will simply write $K^{\star,2}$ by omitting the parameters, if this does not lead to confusion. In the same way that the operator $K^\star_{\alpha,t}$ is the kernel of the well-known Brownian representation of fBm integration, our operator $K^{\star,2}_{\alpha,\beta}$ satisfies, for any test function $f \in \left(K^{\star,2}\right)^{-1}\left(L^2[0,T]\right)$,

$$\int_0^s\int_0^t W^{\alpha,\beta}(dq,dr)\,f(q,r) = \int_{q=0}^s\int_{r=0}^t \left[K^{\star,2}_{\alpha,\beta,t,s}f\right](q,r)\,dW_q dW_r,$$

where the integrals on either side are of Wiener type.

2.2 Extended integral

The extended integral can be defined as in the one-dimensional situation in [2] or [7]: we introduce the Hilbert space

$$\mathcal{H}^{(2),\prime} = \left(K^{\star,2,adj}K^{\star,2}\right)^{-1}\left(L^2(0,1)\right)$$

where $K^{\star,2,adj}$ is the adjoint of the operator $K^{\star,2}$, and we construct the Malliavin derivative D as above, relative to the new space $\mathcal{H}^{(2),\prime}$ instead of $\mathcal{H}^{(2)}$. Since $\mathcal{H}^{(2),\prime}$ is smaller than $\mathcal{H}^{(2)}$, this definition is immediate. We will say that a square integrable process U belongs to the extended domain of the divergence operator ($U \in Dom^{\star}(\delta)$ if there exists a random variable (that we shall also denote by $\delta(U)$) such that

$$E(F\delta(U)) = \int\int_T E\left[U_{s,t}\left(K^{\star,2,adj}K^{\star,2}D_{\cdot,\cdot}F\right)(s,t)\right]dsdt \qquad \forall F \in \mathcal{S}_{\mathcal{H}^{(2),\prime}}. \tag{2}$$

This way, shifting the adjoint back onto F, we see that the 'new' extended integral restricted to $Dom(\delta)$ coincides with the standard Skorohod integral. In the sequel we will simply write $\mathcal{H}, \mathcal{H}'$ instead of $\mathcal{H}^{(2)}, \mathcal{H}^{(2),\prime}$. The reader may consult [7] for a proof that $\mathcal{H}'$ is not restricted to constant random variables. In fact, [7] established that $\mathcal{H}'$ is rich enough to guarantee that the above definition of $\delta(U)$ defines a unique random variable in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ if $\mathcal{F}$ is the sigma-field generated by $W^{\alpha,\beta}$. This uniqueness is usually called the *determining class* property of $\mathcal{S}_{\mathcal{H}'}$ for δ. It is remarkable to note that, now that our operator $K^{\star,2}$ is defined, and the existence and determining class properties of $\mathcal{S}_{\mathcal{H}'}$ are established, there will no longer be any reference to the fractional calculus. We contend that Skorohod integration, extended or not, should not require the use of fractional calculus: one should only have to specify how the kernel $K^{\star}$ is defined, by any analytic method, which may or may not refer to factional calculus, and show the space $\mathcal{S}_{\mathcal{H}'}$ of test random variables F is a determining class. That such characteristics are sufficient for developing a full stochastic calculus is the underlying argument in [7], which proves it in the single-parameter case for a wide class of Volterra-type processes which span the fractional Brownian scale and go beyond. The present article shows that the same program can be achieved for two-parameter processes.

It is also possible to characterize the extended domain $Dom^{\star}(\delta)$ using the multiple stochastic integrals. We recall Theorem 3.2 of [6] (proved in the one-parameter case; but it can be immediately extended to the two-parameter case). Let u be a square integrable process having the chaos representation

$$u(s,t) = \sum_{n\geq 0} I_n(f_n(\cdot,(s,t)))$$

where I_n denotes the multiple integral of order n with respect to $W^{\alpha,\beta}$ and $f_n \in \mathcal{H}^{\otimes n} \otimes L^2(T)$ is symmetric in the first n pairs of variables. Then $u \in Dom^{\star}(\delta)$ if and only if $\tilde{f}_n$ (the symmetrization of f_n in all variables) belongs to $\mathcal{H}^{\otimes n+1}$ and

$$\sum_{n\geq 0}(n+1)!|\tilde{f}_n|^2_{\mathcal{H}^{\otimes n+1}} < \infty. \tag{3}$$

In this case $\delta(u) = \sum_{n\geq 0} I_{n+1}(f_n)$.

2.3 Specific double integral

We need to introduce a double Skorohod integral that appears in the expression of the Itô formula for the fractional Brownian sheet. To motivate this definition, let us briefly recall some elements of the regular case when the parameters α and β are bigger than $\frac{1}{2}$. In this situation we have the following decomposition formula for $(W^{\alpha,\beta})^2$

$$(W^{\alpha,\beta}_{s,t})^2 = 2\int_0^s \int_0^t W^{\alpha,\beta}_{u,v} dW^{\alpha,\beta}_{u,v} + 2\tilde{M}_{s,t} + s^{2\alpha}t^{2\beta} \tag{4}$$

where the process $\tilde{M}_{s,t}$ is defined as the limit

$$\tilde{M}_{s,t} = L^2(\Omega) - \lim_{|\pi|\to 0} \delta^{(2)} \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 1_{[s_i,s_{i+1}]\times[0,t_j]}(\cdot) 1_{[0,s_i]\times[t_j,t_{j+1}]}(*) \right) \tag{5}$$

where $\delta^{(2)}$ denotes the (standard) double Skorohod integral and the above limit exists. Moreover, the Itô formula for the fractional sheet contains a "specific" sheet integral $\int_0^s \int_0^t f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v}$ which is defined as

$$\int_0^s \int_0^t f''(W^{\alpha,\beta})_{u,v} d\tilde{M}_{u,v} = L^2(\Omega) - \lim_{|\pi|\to 0} \delta^{(2)} \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{[s_i,s_{i+1}]\times[0,t_j]}(\cdot) 1_{[0,s_i]\times[t_j,t_{j+1}]}(*) \right). \tag{6}$$

The fact that the parameters α and β are supposed to be bigger than $1/2$ plays an essential role in the proof of the convergence of the sequences from the right side of (5) and (6). Therefore, for small parameters, the integral $d\tilde{M}$ should be understood in an extended way. Nevertheless, the intuitive interpretation of $\tilde{M}$ is that

$$d\tilde{M}_{s,t} = d_s W_{s,t} \cdot d_t W_{s,t}$$

where for example $d_s W_{s,t}$ denotes the differential of the fBm $s \mapsto W_{s,t}$ when t is fixed. Accordingly we can formally write

$$\begin{aligned}
&\int_0^1 \int_0^1 g\left(W^{\alpha,\beta}_{u,v}\right) d\tilde{M}_{u,v} \\
&= \int_{u'=0}^{u'=1} \int_{v=0}^{v=1} \left(\int_{u=0}^{u=1} \int_{v'=0}^{v'=1} g\left(W^{\alpha,\beta}_{u,v}\right) 1_{[0,u]}(u') 1_{[0,v]}(v') W^{\alpha,\beta}(du,dv') \right) W^{\alpha,\beta}(du',dv) \\
&= \int_{u=0}^{u=1} \int_{v'=0}^{v'=1} \left(\int_{u'=0}^{u} \int_{v=v'}^{1} g\left(W^{\alpha,\beta}_{u,v}\right) W^{\alpha,\beta}(du',dv) \right) W^{\alpha,\beta}(du,dv')
\end{aligned}$$

and this shows the the integral $\int\int g(W)d\tilde{M}$ should be interpreted as an iterated integral. We now only need to define the iterated integral as an extended divergence integral.

Definition 1 *Let $U \in L^2(T \times T \times \Omega)$. We say that the process U belongs to the extended domain of $\delta^{(2)}$ ($U \in Dom^\star\left(\delta^{(2)}\right)$) if there exists a random variable $\delta^{(2)}(U) \in L^2(\Omega)$ such that , for every smooth random* *variable* $F \in \mathcal{S}_{\mathcal{H}^{(2)},'}$ *it holds that*

$$E\left(F\delta^{(2)}(U)\right) = \int\int_{(u,v)\in T}\int\int_{(u',v')\in T} E\left[U_{(u,v),(u',v')}K^{\star,2,adj}K^{\star 2}\left\{D_{(\cdot,\cdot)}\left[K^{\star,2,adj}K^{\star,2}D_{(*,*)}F\right](u,v)\right\}(u',v')\right] dudvdu'dv'. \tag{7}$$

We will also write

$$\delta^{(2)}(U) := \int\int_T \left(\int\int_T U_{(u,v),(u',v')}dW^{\alpha,\beta}(u,v)\right) W^{\alpha,\beta}(u',v').$$

Remark 1 *Since the action of the operator* $L = K^{\star,2,adj}K^{\star 2}$ *is deterministic, we have for every smooth random variable* F*,*

$$K^{\star,2,adj}K^{\star 2}\left\{D_{(\cdot,\cdot)}\left[K^{\star,2,adj}K^{\star 2}D_{(*,*)}F\right](u,v)\right\}(u',v') = (L\otimes L)(D^{(2)}_{(\cdot,\cdot),(*,*')}F)\left((u',v'),(u,v)\right)$$

where $D^{(2)}$ *is the second iterated Malliavin derivative. Therefore, relation (7) can be written as*

$$E\left(F\delta^{(2)}(U)\right) = \int_{T^2} E\left[U_{(u,v),(u',v')}(L\otimes L)\left(D^{(2)}_{(\cdot,\cdot),(*,*')}F\right)(u',v'),(u,v)\right] dudvdu'dv'.$$

3 Main result

In this section we derive the Itô formula for the fractional Brownian sheet for any Hurst parameter, by using the technique introduced in [2] and [7] based on the extended Skorohod integral. However, there is one complication in our situation which was not present in the one-parameter settings of [2] and [7]. In these two works, the Itô formula can be considered as an equality between two terms: an extended Skorohod integral I, and the sum S of a Riemann integral and a deterministic function of the underlying process. The idea is then only to show that $S = I$ by proving that S satisfies the definition of I in the extended Skorohod sense; indeed, we then obtain the existence of the Skorohod integral and the Itô formula simultaneously. In our situation, we cannot proceed this way directly because we will have in our Itô formula not one but four Skorohod integrals with respect to different differentials. Therefore, as a preliminary step, we must show that three of the four extended Skorohod integrals exist a-priori, so that we may use their definition to prove the final result. Throughout, we use the generic notation $\underline{t}$ for a pair $(s,t) \in T = [0,1]^2$. For convenience's sake, we also use the abusive notation $h(\underline{t})1^{\otimes n}_{[0,\underline{t}]}(\cdot)$ for the function defined on T^{n+1} by the map

$$(\underline{t},\underline{t}_1,\underline{t}_2,\cdots,\underline{t}_n) \mapsto h(\underline{t})1^{\otimes n}_{[0,\underline{t}]}(\underline{t}_1,\underline{t}_2,\cdots,\underline{t}_n). \tag{8}$$

We start with the following result.

Lemma 1 *Let $f \in C^{\infty}(\mathbb{R})$ satisfying (10) and put $h(\underline{t}) = E\left(f\left(W_{\underline{t}}^{\alpha,\beta}\right)\right)$. Then it holds that*

$$h(\underline{t})1_{[0,\underline{t}]}^{\otimes n}(\cdot) \in \left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)} \qquad \text{for every } n \geq 1.$$

and there exist an integer N large enough such that if $n \geq N$ we have

$$\|h(\underline{t})1_{[0,\underline{t}]}^{\otimes n}\|^2_{\left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)}} \leq \frac{C}{n}. \tag{9}$$

Proof: Let us prove first the result when f is a polynomial function; moreover, without loss of generality, let $f(x) = x^p$, where p is an even integer (for odd integers, h is null). Then with $\underline{t} = (t,s)$, we have $h(\underline{t}) = c_p t^{\alpha p} s^{\beta p}$. Since $\mathcal{H}^{(2)}$ is the tensor product Hilbert space $\mathcal{H}^{(2)} = \mathcal{H}_\alpha \otimes \mathcal{H}_\beta$ (where $\mathcal{H}_\beta := \mathcal{H}$ is the canonical space of the one-parameter fBm B^β with Hurst parameter β), it suffices to prove, using the one-parameter version of the abusive notation (8), that

$$t^{\alpha p}1_{[0,t]}(\cdot) \in \mathcal{H}^{\otimes(n+1)}$$

or, equivalently, $K^{\star,n+1}\left(t^{\alpha p}1_{[0,t]}^{\otimes n}(t_1,\ldots,t_n)\right) \in L^2([0,1]^{n+1})$ (where $K^{\star,n}$ is the n-fold tensor product operator of $K^{\star,1}$, and we use the abusive notation of naming a function by its value). Using the definition of the operator $K^{\star,n+1}$ it is not difficult to observe that

$$\begin{aligned}
&\|K^{\star,n+1}\left(t^{\alpha p}1_{[0,t]}^{\otimes n}(t_1,\ldots,t_n\right)\|^2_{L^2([0,1]^{n+1})} \\
&= \|K^{\star,1}\left[t^{\alpha p}\|K^{\star,n}1_{[0,t]}^{\otimes n}(t_1,\ldots,t_n)\|^2_{L^2([0,1]^n)}\right]\|^2_{L^2([0,1])} \\
&= \|K^{\star,1}\left[t^{\alpha p}\|K^{\star,1}1_{[0,t]}(\cdot)\|^{2n}_{L^2([0,1])}\right]\|^2_{L^2([0,1])}
\end{aligned}$$

Note first that

$$\|K^{\star,1}1_{[0,t]}(\cdot)\|^2_{L^2([0,1])} = E\left(B_t^2\right) = t^{2\alpha}.$$

Consequently, we only need to prove that the function $t^{\alpha(p+2n)}$ has a finite norm in $\mathcal{H}$. To argue this, let us refer to Proposition 7 in [2] which states that if a process u is in $Dom^\star(\delta)$ such that $E[u.] \in L^2(\mathbb{R})$, then $E[u.]$ is in $\mathcal{H}$. But $t^{\alpha(p+2n)}$ is equal to $E\left(B^{n+\frac{p}{2}}(t)\right)$ which belongs to $Dom^\star(\delta)$ due to Lemma 9 in [2]. The inequality (9) can be proved using e.g. the fact that for fixed α, there exists N large enough such that the function $t^{\alpha(N+p/2)}$ is Lipschitz. Then it can be seen by a straightforward calculation that the $L^2[0,1]$-norm of $K^{\star,1}t^{\alpha(N+p/2)}$ (thus the $\mathcal{H}$-norm of $t^{\alpha(N+2p)}$) is bounded by C/N, and that this bound is uniform in p. The reader may also refer to the calculations in Section 5, which are valid in all cases including the fractional Brownian scale, for a proof of estimates such as (9).

The general case when f is C^∞ follows by a density argument. Let us only point out the main idea. Now

$$h(\underline{t}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}} f(xt^\alpha s^\beta)dx.$$

The key point of the proof is to show that the function $f(xt^\alpha)t^{2\alpha n}$ is in $\mathcal{H}$ and this can be seen, for example, by using a polynomial approximation of f, the definition of the operator $K^{\star,1}$ and the dominated convergence theorem. Condition (10) assures the existence of the integral with respect to dx. ∎

We may now prove our preliminary existence result.

Proposition 1 *There exists $a > 0$ depending only on α and β such that for any $f \in C^\infty(\mathbb{R})$ such that f and all its derivatives satisfy the condition*

$$|f(x)| \leq M \exp\left(ax^2\right) \tag{10}$$

for $|x|$ large enough, where M is a positive constant, we have

$$1_{[0,s_0]\times[0,t_0]}(s,t) f(W^{\alpha,\beta}_{s,t}) \in Dom^\star(\delta) \text{ for every } s_0, t_0 \in [0,1].$$

Proof: We will assume that $s_0 = t_0 = 1$; the general case is analogous. Using Stroock's formula (see [?]) we get,

$$\begin{aligned} f(W^{\alpha,\beta}_{s,t}) &= \sum_{n\geq 0} \frac{1}{n!} I_n \left[D^{(n)} \left(f^{(n)}(W^{\alpha,\beta}_{\underline{t}}) \right) \right] \\ &= \sum_{n\geq 0} \frac{1}{n!} E\left(f^{(n)}(W^{\alpha,\beta}_{\underline{t}}) \right) I_n \left(1^{\otimes n}_{[0,\underline{t}]}(\cdot) \right) \\ &= \sum_{n\geq 0} I_n \left(g_n(\cdot, \underline{t}) \right) \end{aligned}$$

where

$$g_n(\cdot, \underline{t}) = \frac{1}{n!} E\left(f^{(n)}(W^{\alpha,\beta}_{\underline{t}}) \right) 1^{\otimes n}_{[0,\underline{t}]}(\cdot).$$

Here '$\cdot$' represents n variables. Let us denote by $\tilde{g_n}$ the symmetrization of g in $n+1$ variables. We need to show that

$$\tilde{g_n} \in \left(\mathcal{H}^{(2)} \right)^{\otimes(n+1)} \tag{11}$$

and

$$\sum_{n\geq 0} (n+1)! \|\tilde{g_n}\|^2_{\left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)}} < \infty. \tag{12}$$

First, observe that (11) holds due to Lemma 1. Also, we have that

$$\tilde{g_n}(\underline{t}_1, \ldots, \underline{t}_{n+1}) = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} h(\underline{t}_i) 1_{[0,\underline{t}_i]}(\hat{t}_i)$$

where $\hat{t}_i$ is the vector $(\underline{t}_1, \ldots, \underline{t}_{n+1})$ with $\underline{t}_i$ missing and $h(\underline{t})$ is the function $h(\underline{t}) = E\left(f^{(n)}(W^{\alpha,\beta}_{\underline{t}}) \right)$. To check (12), we can write using the Lemma 1, that for some N large enough

$$\begin{aligned} \sum_{n\geq N} (n+1)! \|\tilde{g_n}\|^2_{\left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)}} &= \sum_{n\geq N} \frac{1}{(n+1)!} \| \sum_{i=0}^{n+1} h(\underline{t}_i) 1_{[0,\underline{t}_i]}(\hat{t}_i) \|^2_{\left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)}} \\ &\leq \sum_{n\geq N} \frac{2^{n+1}}{(n+1)!} \sum_{i=0}^{n+1} \| h(\underline{t}_i) 1_{[0,\underline{t}_i]}(\hat{t}_i) \|^2_{\left(\mathcal{H}^{(2)}\right)^{\otimes(n+1)}} \\ &\leq \sum_{n\geq N} \frac{C 2^{n+1}}{n n!} < \infty. \end{aligned}$$

■

Denote by H_n the n -th Hermite polynomial

$$H_0(x) := 1 \text{ and } H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right).$$

and recall the basic properties

$$DH_n(W^{\alpha,\beta}(\varphi)) = H_{n-1}(W^{\alpha,\beta}(\varphi))\varphi \tag{13}$$

for every $\varphi \in \mathcal{H}^{(2),\prime}$, and

$$\delta\left(H_{k-1}(W^{\alpha,\beta}(\varphi))\varphi\right) = kH_k(W^{\alpha,\beta}(\varphi)) \tag{14}$$

We state our main result.

Theorem 1 *Let $f \in C^\infty(\mathbb{R})$ such that f and all its derivatives satisfy (10). Then $f'(W^{\alpha,\beta})1_{[0,s]\times[0,t]} \in Dom^\star(\delta)$, and $f''(W^{\alpha,\beta}_{u,v})1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v') \in Dom^\star(\delta^{(2)})$ and we have the following Itô formula for the fractional Brownian sheet:*

$$\begin{aligned} f(W^{\alpha,\beta}_{s,t}) &= f(0) + \int_0^s \int_0^t f'(W^{\alpha,\beta}_{u,v}) dW^{\alpha,\beta}_{u,v} \\ &+ 2\alpha\beta \int_0^s \int_0^t f''(W^{\alpha,\beta}_{u,v}) u^{2\alpha-1} v^{2\beta-1} dv du + \int_0^s \int_0^t f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v} \\ &+ \alpha \int_0^s \int_0^t f'''(W^{\alpha,\beta}_{u,v}) u^{2\alpha-1} v^{2\beta} d_v W^{\alpha,\beta}_{u,v} du + \beta \int_0^s \int_0^t f'''(W^{\alpha,\beta}_{u,v}) u^{2\alpha} v^{2\beta-1} dv d_u W^{\alpha,\beta}_{u,v} du \\ &+ \alpha\beta \int_0^s \int_0^t f^{iv}(W^{\alpha,\beta}_{u,v}) u^{4\alpha-1} v^{4\beta-1} dv du \end{aligned}$$

where, by definition,

$$\int_0^s \int_0^t f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v} = \iint_T \left(\iint_T f''(W^{\alpha,\beta}_{u,v}) 1_{[0,s]}(u) 1_{[0,t]}(v) 1_{[0,u]}(u') 1_{[0,v]}(v') W^{\alpha,\beta}(du', dv) \right) W^{\alpha,\beta}(du, dv') \tag{15}$$

and we recall that $d_u W^{\alpha,\beta}_{u,v}$ denotes the Skorohod differential of the one-parameter fractional Brownian motion $u \to W^{\alpha,\beta}_{u,v}$.

Proof: By Proposition 1 it holds that $f'(W^{\alpha,\beta})1_{[0,s]\times[0,t]} \in Dom^\star(\delta)$ for every s, t. Similar arguments allow to show the integrability of the integrand for the other two Skorohod integrand in the right side excepting the one involving $\tilde{M}$. The existence of the stochastic integral with respect to $\tilde{M}$ in the Itô formula follows, by definition of $\tilde{M}$, from the second statement in the theorem. This second statement, on membership in $Dom^\star\left(\delta^{(2)}\right)$, is not, strictly speaking, contained in Proposition 1, but its proof is a trivial generalization to double integrals of the proof of Proposition 1. We omit all details. The existence of the remaining two stochastic integrals in the Itô formula follows trivially from existence results in [2], since these stochastic integrals are, by Fubini, with respect to one-parameter fBm's. Now using the definition of the extended divergence integral, it suffices to show, invoking only simple random variables of the form $F = H_n\left(W^{\alpha,\beta}(\varphi)\right)$, since they are dense in $L^2(\Omega)$, that

$$\begin{aligned}
&\int\int_{(u,v')\in T}\int_{(u',v)\in T} 1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v') \\
&\times E\left[H_{n-2}(W^{\alpha,\beta}(\varphi))f''(W^{\alpha,\beta}_{u,v})\right]\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v')\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u',v)dudu'dvdv' \\
&= E\left\{\left[f(W^{\alpha,\beta}_{s,t}) - f(0) - 2\alpha\beta\int_0^s\int_0^t f''(W^{\alpha,\beta}_{u,v})u^{2\alpha-1}v^{2\beta-1}dvdu-\right.\right. \\
&-\alpha\int_0^s\int_0^t f'''(W^{\alpha,\beta}_{u,v})u^{2\alpha-1}v^{2\beta}d_vW^{\alpha,\beta}_{u,v}du - \beta\int_0^s\int_0^t f'''(W^{\alpha,\beta}_{u,v})u^{2\alpha}v^{2\beta-1}dvd_uW^{\alpha,\beta}_{u,v}du \\
&\left.\left.-\alpha\beta\int_0^s\int_0^t f^{iv}(W^{\alpha,\beta}_{u,v})u^{4\alpha-1}v^{4\beta-1}dvdu\right]H_n\left(W^{\alpha,\beta}(\varphi)\right)\right\} \\
&- E\int_0^s\int_0^t f'(W^{\alpha,\beta}_{u,v})H_{n-1}(W^{\alpha,\beta}(\varphi))\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v)dvdu. \qquad (16)
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right] &= \frac{\partial^2}{\partial s\partial t}\int_{\mathbb{R}} p\left(s^{2\alpha}t^{2\beta},y\right)f^{(n)}(y)dy \\
&= \frac{\partial}{\partial s}\int_{\mathbb{R}}\frac{\partial}{\partial\sigma}p\left(s^{2\alpha}t^{2\beta},y\right)2\beta t^{2\beta-1}s^{2\alpha}f^{(n)}(y)dy \\
&= 4\alpha\beta s^{4\alpha-1}t^{4\beta-1}\int_{\mathbb{R}}\frac{\partial^2}{\partial\sigma^2}p\left(s^{2\alpha}t^{2\beta},y\right)f^{(n)}(y)dy \\
&+ 4\alpha\beta s^{2\alpha-1}t^{2\beta-1}\int_{\mathbb{R}}\frac{\partial}{\partial\sigma}p\left(s^{2\alpha}t^{2\beta},y\right)f^{(n)}(y)dy.
\end{aligned}$$

Using the integration by parts and the relation

$$\frac{\partial p}{\partial\sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$$

we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right] &= \alpha\beta s^{4\alpha-1}t^{4\beta-1}E\left[f^{(n+4)}(W^{\alpha,\beta}_{s,t})\right] \\
&+ 2\alpha\beta s^{2\alpha-1}t^{2\beta-1}E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t})\right] \qquad (17)
\end{aligned}$$

and that proves (16) in the case $n = 0$ (i.e. the case when the test r.v. is $F = 1$). Note also that

$$\frac{\partial}{\partial s}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right] = \alpha t^{2\beta}s^{2\alpha-1}E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t})\right] \qquad (18)$$

and

$$\frac{\partial}{\partial t}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right] = \beta t^{2\beta-1}s^{2\alpha}E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t})\right]. \qquad (19)$$

We compute now

$$\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}\left(E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right]\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n\right) \\
&= \left(\frac{\partial^2}{\partial s\partial t}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right]\right)\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n + \frac{\partial}{\partial s}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right]\frac{\partial}{\partial t}\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n \\
&+ \frac{\partial}{\partial t}E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right]\frac{\partial}{\partial s}\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n + E\left[f^{(n)}(W^{\alpha,\beta}_{s,t})\right]\frac{\partial^2}{\partial s\partial t}\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n. \qquad (20)
\end{aligned}$$

On the other hand, using the identity (this is a consequence of the fractional calculus; see [2] for the one-dimensional case)

$$\int_0^s \int_0^t \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,v) dv du = \langle 1_{[0,s]\times[0,t]}, \varphi \rangle$$

we obtain the relations

$$\frac{\partial}{\partial s} \langle 1_{[0,s]\times[0,t]}, \varphi \rangle = \int_0^t \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(s,v) dv \tag{21}$$

and

$$\frac{\partial}{\partial t} \langle 1_{[0,s]\times[0,t]}, \varphi \rangle = \int_0^s \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,t) du. \tag{22}$$

By combining relations (21), (22) and (18) with (20), we get

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t} \left(E\left[f^{(n)}(W^{\alpha,\beta}_{s,t}) \right] \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^n \right) \\
&= \left\{ \alpha\beta s^{4\alpha-1} t^{4\beta-1} E\left[f^{(n+4)}(W^{\alpha,\beta}_{s,t}) \right] + 2\alpha\beta s^{2\alpha-1} t^{2\beta-1} E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t}) \right] \right\} \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^n \\
&+ \alpha s^{2\alpha-1} t^{2\beta} E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t}) \right] n \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^{n-1} \int_0^t \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,t) du \\
&+ \beta s^{2\alpha} t^{2\beta-1} E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t}) \right] n \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^{n-1} \int_0^s \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(s,v) dv \\
&+ E\left[f^{(n)}(W^{\alpha,\beta}_{s,t}) \right] n(n-1) \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^{n-2} \left(\int_0^s \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(s,v) dv \right) \left(\int_0^t \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,t) du \right) \\
&+ E\left[f^{(n)}(W^{\alpha,\beta}_{s,t}) \right] n \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^{n-1} \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(s,t).
\end{aligned}$$

Therefore

$$\begin{aligned}
&E\left[f^{(n)}(W^{\alpha,\beta}_{s,t}) \right] \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^n \\
&= 2\alpha\beta E \int_0^s \int_0^t u^{2\alpha-1} v^{2\beta-1} f^{(n+2)}(W^{\alpha,\beta}_{u,v}) \langle 1_{[0,u]\times[0,v]}, \varphi \rangle^n dv du \\
&+ \alpha\beta E \int_0^s \int_0^t u^{4\alpha-1} v^{4\beta-1} f^{(n+4)}(W^{\alpha,\beta}_{u,v}) \langle 1_{[0,u]\times[0,v]}, \varphi \rangle^n dv du \\
&+ \alpha E \int_0^s \int_0^t u^{2\alpha-1} v^{2\beta} f^{(n+2)}(W^{\alpha,\beta}_{u,v}) n \langle 1_{[0,u]\times[0,v]}, \varphi \rangle^{n-1} \left(\int_0^u \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(x,v) dx \right) du dv \\
&+ \text{ its symmetric term}
\end{aligned}$$

$$\begin{aligned}
&+ E \int_0^s \int_0^t f^{(n)}(W^{\alpha,\beta}_{u,v}) n(n-1) \langle 1_{[0,u]\times[0,v]}, \varphi \rangle^{n-2} du dv \\
&\times \left(\int_0^u \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(x,v) dx \right) \left(\int_0^v \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,y) dy \right) \\
&+ E \int_0^s \int_0^t f^{(n)}(W^{\alpha,\beta}_{u,v}) n \langle 1_{[0,u]\times[0,v]}, \varphi \rangle^{n-1} f^{(n)}(W^{\alpha,\beta}_{u,v}) \left(K^{\star,2,adj} K^{\star,2} \varphi\right)(u,v) du dv.
\end{aligned} \tag{23}$$

By iterating the duality relation (2) and using (13) and (14), we can prove

$$E\left[f^{(n)}(W^{\alpha,\beta}_{s,t}) \right] \langle 1_{[0,s]\times[0,t]}, \varphi \rangle^n = n! E\left[f(W^{\alpha,\beta}_{s,t}) H_n(W^{\alpha,\beta}(\varphi)) \right], \tag{24}$$

$$E\left[f^{(n+2)}(W^{\alpha,\beta}_{s,t})\right]\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n = n!E\left[f''(W^{\alpha,\beta}_{s,t})H_n(W^{\alpha,\beta}(\varphi))\right], \tag{25}$$

$$E\left[f^{(n+4)}(W^{\alpha,\beta}_{s,t})\right]\langle 1_{[0,s]\times[0,t]},\varphi\rangle^n = n!E\left[f^{iv}(W^{\alpha,\beta}_{s,t})H_n(W^{\alpha,\beta}(\varphi))\right] \tag{26}$$

and

$$\begin{aligned}&E\int_0^s\int_0^t f^{(n)}(W^{\alpha,\beta}_{u,v})n\langle 1_{[0,u]\times[0,v]},\varphi\rangle^{n-1}f^{(n)}(W^{\alpha,\beta}_{u,v})\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v)dudv\\&= n!E\int_0^s\int_0^t f'(W^{\alpha,\beta}_{u,v})H_{n-1}(W^{\alpha,\beta}(\varphi))\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v)dvdu\end{aligned} \tag{27}$$

Taking into account relations (23), (24), (25), (26) and (27), we only need to show that

$$\begin{aligned}&\alpha E\int_0^s\int_0^t u^{2\alpha-1}v^{2\beta}f^{(n+2)}(W^{\alpha,\beta}_{u,v})n\langle 1_{[0,u]\times[0,v]},\varphi\rangle^{n-1}\left(\int_0^u\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(x,v)dx\right)dudv\\&= \alpha E\left[\int_0^s\int_0^t f'''(W^{\alpha,\beta}_{u,v})u^{2\alpha-1}v^{2\beta}d_vW^{\alpha,\beta}_{u,v}du\right]H_n(W^{\alpha,\beta}(\varphi))\end{aligned} \tag{28}$$

(and an analogue for its symmetric term), and

$$\begin{aligned}&E\int_0^s\int_0^t f^{(n)}(W^{\alpha,\beta}_{u,v})n(n-1)\langle 1_{[0,u]\times[0,v]},\varphi\rangle^{n-2}dudv\\&\times\left(\int_0^u\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(x,v)dx\right)\left(\int_0^v\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,y)dy\right)\\&= \int\int_{(u,v')\in T}\int_{(u',v)\in T}1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v')\\&\times E\left[H_{n-2}(W^{\alpha,\beta}(\varphi))f''(W^{\alpha,\beta}_{u,v})\right]\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v')\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u',v)\end{aligned} \tag{29}$$

To prove the equality (29), we will use the duality relation (7) from Definition (1). We have

$$\begin{aligned}&= \int\int_{(u,v')\in T}\int_{(u',v)\in T}1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v')\\&\times E\left[H_{n-2}(W^{\alpha,\beta}(\varphi))f''(W^{\alpha,\beta}_{u,v})\right]\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,v')\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u',v)\\&= \int_0^s\int_0^t dudvE\left[H_{n-2}(W^{\alpha,\beta}(\varphi))f''(W^{\alpha,\beta}_{u,v})\right]\\&\times\left(\int_0^u\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(x,v)dx\right)\left(\int_0^v\left(K^{\star,2,adj}K^{\star,2}\varphi\right)(u,y)dy\right)\end{aligned}$$

and relation (29) follows since similar arguments as above imply that

$$\begin{aligned}&E\int_0^s\int_0^t f^{(n)}(W^{\alpha,\beta}_{u,v})n(n-1)\langle 1_{[0,u]\times[0,v]},\varphi\rangle^{n-2}dudv\\&= n!\int_0^s\int_0^t dudvE\left[H_{n-2}(W^{\alpha,\beta}(\varphi))f''(W^{\alpha,\beta}_{u,v})\right].\end{aligned}$$

Relation (28) is established similarly. ■

4 Local time

We give a brief discussion about the integral representation of the local time of a fractional Brownian sheet. In general, there are two methods to define local times for a stochastic process X: the first one is Berman's approach ([1]) based on direct calculations and Fourier analysis, where the local time is defined as the density of the occupation measure $\lambda_t(A) = \int_A 1_A(X_s)ds$; the second method is the Tanaka formula (for processes X for which such a formula can be written) where the local time appears as the last term in the decomposition of $|X_s - a|$. We have the following situation:

- for the one-dimensional Brownian motion the two approaches gives the same local time;
- for the Brownian sheet W the situation changes; we have two different local times, the 'Tanaka formula' local time being the density of the occupation measure (see [3], Chapter 6)

$$\lambda_{s,t}(A) = \int_0^s \int_0^t 1_A(W_{u,v})uvdudv;$$

- concerning the fractional Brownian motion B^H, the difference between the two approaches appears even in the one-parameter case: the Tanaka formula, valid for Skorohod integration, implies the existence of a local time associated with the weighted occupation measure

$$\lambda_t(A) = H \int_0^t 1_A(B_s^H)s^{2H-1}ds.$$

Therefore, since our framework is that of Skorohod integration, it is natural to introduce the local time $(L^a_{s,t})_{(s,t)\in T, a\in\mathbb{R}}$ of the fractional Brownian sheet as the density of the occupation measure

$$\lambda_t(A) = \alpha\beta \int_0^s \int_0^t 1_A(W^{\alpha,\beta}_{u,v})u^{4\alpha-1}v^{4\beta-1}dudv.$$

A chaos expansion argument (see e.g. [4]) can be used to show the existence of the local time. It is also clear that the techniques of the regular case $\alpha, \beta > \frac{1}{2}$ (see [10]) could be adapted to the singular case to obtain a Tanaka-type formula. We will only state the result; the proof is left to the reader.

Proposition 2 *For every $(s,t) \in T$ and $a \in \mathbb{R}$, it holds that*

$$\begin{aligned} L^a_{s,t} &= \frac{1}{6}\left|W^{\alpha,\beta}_{s,t} - a\right|\left(W^{\alpha,\beta}_{s,t} - a\right)^2 - \frac{1}{2}\int_0^s \int_0^t \left|W^{\alpha,\beta}_{u,v} - a\right|\left(W^{\alpha,\beta}_{u,v} - a\right)dW^{\alpha,\beta}_{u,v} \\ &- 2\alpha\beta \int_0^s \int_0^t \left|W^{\alpha,\beta}_{u,v} - a\right| u^{2\alpha-1}v^{2\beta-1}dvdu - \int_0^s \int_0^t \left|W^{\alpha,\beta}_{u,v} - a\right| d\tilde{M}_{u,v} \\ &- \alpha \int_0^s \int_0^t sign(W^{\alpha,\beta}_{u,v})u^{2\alpha-1}v^{2\beta}d_vW^{\alpha,\beta}_{u,v}du - \beta \int_0^s \int_0^t sign(W^{\alpha,\beta}_{u,v})u^{2\alpha}v^{2\beta-1}dvd_uW^{\alpha,\beta}_{u,v} \end{aligned}$$

where the integrals with respect to the differentials $dW^{\alpha,\beta}_{u,v}$, $d\tilde{M}_{u,v}$ (see (15)), $d_uW^{\alpha,\beta}_{u,v}$ and $d_vW^{\alpha,\beta}_{u,v}$ are in the extended sense.

5 Two-parameter calculus beyond the fractional scale

5.1 Introduction and definitions

In this last section, we consider the analogue of the Theorem 1 and Proposition 2 when one does not wish to restrict oneself to the fractional scale. The one-parameter Skorohod stochastic integral for Gaussian processes beyond the fractional scale was introduced in [7], as was alluded to earlier. We summarize the construction therein by generalizing to our two-paramter setting. Because no fundamentally new ideas are required in the passage from one to two parameters, we state the results without proof. Assume that γ,g are two smooth strictly increasing functions on $(0,1]$, such that $\lim_{0+} g = \lim_{0+} \gamma = 0$, and for r near 0, we have $\gamma^2(r) \gg r$ and $g^2(r) \gg r$. Then let W be a Wiener process and consider the centered Gaussian field defined by the double Wiener stochastic integral

$$B^{g,\gamma}(s,t) := \int_{r=0}^{t} \int_{q=0}^{s} \varepsilon(t-r)\, h(s-q)\, dW(r)\, dW(q) \tag{30}$$

where

$$\varepsilon(r) := \left(\frac{d(\gamma^2)}{dr} \right)^{1/2}, \quad h(r) := \left(\frac{d(g^2)}{dr} \right)^{1/2}.$$

The random field $B^{g,\gamma}$ is very close to the fractional Brownian sheet $W^{\alpha,\beta}$ if we take $\gamma(r) = r^{\alpha}$ and $g(r) = r^{\beta}$: the difference between the two fields is a field of bounded variation, and both fields are centered and have the α- and β-self-similar properties in t and s. But if we take for example $\gamma_{\alpha}(r) = \log^{-\alpha}(1/r)$, and similarly for g, then $B^{\gamma_\alpha,\gamma_\beta}$ is much more irregular than the fractional Brownian sheet. In fact, $B^{\gamma_\alpha,\gamma_\beta}$ is almost-surely locally uniformly continuous in t if and only if $\alpha > 1/2$, and similarly for s and β. If $\alpha \le 1/2$, then although $B^{\gamma_\alpha,\gamma_\beta}$ may still be continuous at any fixed point t, it is not a.s. continuous on any interval, and is unbounded on any interval. Thus the scale of logarithmic regularity defined by the example $(\gamma_\alpha)_{\alpha>0}$ yields a spectrum of uniformly continuous and unbounded Gaussian fields. Nevertheless, the theory of Skorohod integration with respect to $B^{g,\gamma}$ can be developed equally easily for $B^{\gamma_\alpha,\gamma_\beta}$ as for $B^{r^{\alpha},r^{\beta}}$, or for $W^{\alpha,\beta}$, by following the arguments in [7]. In the case $\alpha \le 1/2$ or $\beta \le 1/2$, however, one runs into trouble for our two-parameter purposes when one tries to prove Lemma 1. We will see below that we require the following general condition on γ and g.

(G) Assume that both $\varepsilon(r)\, r^{1/2}$ and $h(r)\, r^{1/2}$ are integrable at the origin.

In the logarithmic scale, this condition clearly means that $\alpha, \beta > 1/2$. More generally, one can prove, using the characterization of almost-sure continuity in [11], that Condition (G) is equivalent to requiring that $B^{g,\gamma}$ is almost-surely uniformly continuous. Assuming Condition (G), all the results that hold in Section 2 for $W^{\alpha,\beta}$ also work for $B^{g,\gamma}$ if we just replace the definition of the univariate operator $K^{\star}$ by the following:

$$K_{\gamma}^{\star} f(t) := \left[f(t)\, \varepsilon(1-t) + \int_{t}^{1} (f(s) - f(t))\, \varepsilon'(s-t)\, ds \right]. \tag{31}$$

Accordingly, if $K_\gamma^\star f(\cdot)$ is in $L^2[0,1]$ then we say that $f \in \mathcal{H}^\gamma$, and all the other operators, spaces, and integrals, such as $\mathcal{H}^{2,\prime}_{\gamma,g}$, δ, $\delta^{(2)}$, and their extended domains, are defined based on this new $K_\gamma^\star$. The Itô and Tanaka formulas will also hold, with identical proofs. We refer the reader to the many details in [7], but to be more convincing, we go through the only calculation in the proof of the Itô formula where the relevant quantities appear, as well as some details of what needs to be changed in the proof of Proposition 1.

5.2 Relevant new calculations

First we note that in notation of the proof of Lemma 1, in the case of $B^{g,\gamma}$, we have

$$h(\underline{t}) := E\left[\left(B^{g,\gamma}(s,t)\right)^p\right] = g^2(s)^{p/2}\gamma^2(t)^{p/2}.$$

Then one must check that $\gamma(t)^{p+2n}$ has a finite norm in $\mathcal{H}$. This follows in the same way as the corresponding result for fBm because the Itô formula in [7] proves that $\left(B^\gamma(t)\right)^{p+2n}$ is in $Dom^\star\delta$. We now show that the inequality (9) holds. We only need to show that

$$K_\gamma^\star\left(\gamma^{n+p/2}\right) \in L^2[0,T],$$

and to evaluate the corresponding norm. We calculate, using the definition in (31),

$$K_\gamma^\star\left(\gamma^{n+p/2}\right)(t) = \left(\gamma^{n+p/2}\right)(t)\,\varepsilon(1-t) + \int_t^1 \left[\left(\gamma^{n+p/2}\right)(s) - \left(\gamma^{n+p/2}\right)(t)\right]\varepsilon'(s-t)\,ds.$$

Since γ is bounded, the first term is clearly in $L^2[0,1]$. For the second term, we operate as follows, using the fact that γ is increasing, bounded, and the fact, which we assume as in [7] without loss of generality since $\gamma^2(r) \gg r$, that γ' is decreasing:

$$\begin{aligned}
&\int_t^1 \left[\left(\gamma^{n+p/2}\right)(s) - \left(\gamma^{n+p/2}\right)(t)\right]\varepsilon'(s-t)\,ds \\
&\leq \int_t^1 \gamma^{n+p/2}(s)\sqrt{\gamma'(t)(s-t)}\,\varepsilon'(s-t)\,ds \cdot (n+p/2)\,\gamma^{n-1+p/2}(s) \\
&= \sqrt{\gamma'(t)}\int_t^1 \sqrt{\gamma'(t)(s-t)}\,\varepsilon'(s-t)\,ds \cdot (n+p/2)\,\gamma^{2n-1+p}(s) \\
&\leq (n+p/2)\sqrt{\gamma'(t)}\left(\sup\gamma\right)^{2n-1+p}\int_0^{1-t}\varepsilon'(r)\sqrt{r}\,dr.
\end{aligned}$$

Squaring and integrating in t, we then have for some constant C_γ depending only on γ,

$$\left\|K_\gamma^\star\left(\gamma^{n+p/2}\right)\right\|^2_{L^2[0,1]} \leq \left(n^2+p^2/4\right)\left(C_\gamma\right)^{n+p/2}.$$

This result is slightly less powerful than the conclusion of Lemma 1, since here we have an n in the numerator rather than the denominator, but we can still use this estimation to finish the proof of Proposition 1 under Condition (10). The details are left to the reader.

Now for Itô's formula's proof, with the notation above, the definition of $B^{g,\gamma}$ immediately implies

$$E\left[f^{(n)}\left(B^{g,\gamma}\left(s,t\right)\right)\right]=\int_{\mathbf{R}}p\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)f^{(n)}\left(y\right)dy$$

and therefore

$$\begin{aligned}\frac{\partial^{2}}{\partial s\partial t}E\left[f^{(n)}\left(B^{g,\gamma}\left(s,t\right)\right)\right] &= \frac{\partial}{\partial s}\int_{\mathbf{R}}\frac{\partial p}{\partial\sigma}\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)\left(g^{2}\right)'\left(s\right)\gamma^{2}\left(t\right)f^{(n)}\left(y\right)dy\\
&= \int_{\mathbf{R}}\frac{\partial^{2}p}{\partial\sigma^{2}}\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)\left(g^{2}\right)'\left(s\right)g^{2}\left(s\right)\left(\gamma^{2}\right)'\left(t\right)\gamma^{2}\left(t\right)f^{(n)}\left(y\right)dy\\
&+ \int_{\mathbf{R}}\frac{\partial p}{\partial\sigma}\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)\left(g^{2}\right)'\left(s\right)\left(\gamma^{2}\right)'\left(t\right)f^{(n)}\left(y\right)dy\\
&= \frac{1}{4}\left(g^{4}\right)'\left(s\right)\left(\gamma^{4}\right)'\left(t\right)\int_{\mathbf{R}}\frac{1}{4}\frac{\partial^{4}p}{\partial\sigma^{4}}\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)f^{(n)}\left(y\right)dy\\
&+ \left(g^{2}\right)'\left(s\right)\left(\gamma^{2}\right)'\left(t\right)\int_{\mathbf{R}}\frac{1}{2}\frac{\partial^{2}p}{\partial\sigma^{2}}\left(g^{2}\left(s\right)\gamma^{2}\left(t\right),y\right)f^{(n)}\left(y\right)dy\\
&= \frac{1}{16}\left(g^{4}\right)'\left(s\right)\left(\gamma^{4}\right)'\left(t\right)E\left[f^{(n+4)}\left(B^{g,\gamma}\left(s,t\right)\right)\right]\\
&+ \frac{1}{2}\left(g^{2}\right)'\left(s\right)\left(\gamma^{2}\right)'\left(t\right)E\left[f^{(n+2)}\left(B^{g,\gamma}\left(s,t\right)\right)\right].\end{aligned}$$

Other, easier calculations yield the first derivatives of these expected values. We then immediately see that the only algebraic differences between the fractional case and the general case for $B^{\gamma,g}$ are that $\alpha s^{4\alpha-1}$ is replaced by $(1/16)\left(g^{4}\right)'(s)$, $\alpha s^{2\alpha-1}$ is replaced by $(1/2)\left(g^{2}\right)'(s)$, of course, $s^{2\alpha}$ is replaced by $g^{2}(s)$, and similarly for γ. Thus we can state the following.

Theorem 2 *Let $f\in C^{\infty}(\mathbb{R})$ such that f and all its derivatives satisfy (10). Assume γ and g satisfy Condition (G). Let $B^{g,\gamma}_{u,v}$ be the centered Gaussian field defined on T by (30), with its corresponding Skorohod integration theory based on the operator $K^{\star,(2)}=K^{\star}_{g}\otimes K^{\star}_{\gamma}$ where the factors are defined in (31). Then $f'(B^{g,\gamma})1_{[0,s]\times[0,t]}\in Dom^{\star}(\delta)$, and $f''(B^{g,\gamma}_{u,v})1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v')\in Dom^{\star}(\delta^{(2)})$, and we have the following Itô formula for the general two-parameter Gaussian field of Volterra type:*

$$\begin{aligned}f(B^{g,\gamma}_{s,t}) &= f(0)+\int_{0}^{s}\int_{0}^{t}f'(B^{g,\gamma}_{u,v})dB^{g,\gamma}_{u,v}\\
&+\frac{1}{2}\int_{0}^{s}\int_{0}^{t}f''(B^{g,\gamma}_{u,v})\left(g^{2}\right)'(u)\left(\gamma^{2}\right)'(v)\,dvdu+\int_{0}^{s}\int_{0}^{t}f''(B^{g,\gamma}_{u,v})d\tilde{M}_{u,v}\\
&+\frac{1}{2}\int_{0}^{s}\int_{0}^{t}f'''(W^{\alpha,\beta}_{u,v})\left(g^{2}\right)'(u)\,\gamma^{2}(v)\,d_{v}B^{g,\gamma}_{u,v}du+\frac{1}{2}\int_{0}^{s}\int_{0}^{t}f'''(W^{\alpha,\beta}_{u,v})g^{2}(u)\left(\gamma^{2}\right)'(v)\,dvd_{u}B^{g,\gamma}_{u,v}du\\
&+\frac{1}{16}\int_{0}^{s}\int_{0}^{t}f^{iv}(W^{\alpha,\beta}_{u,v})\left(g^{4}\right)'(s)\left(\gamma^{4}\right)'(s)\,dvdu\end{aligned}$$

where, by definition,

$$\int_{0}^{s}\int_{0}^{t}f''(B^{g,\gamma}_{u,v})d\tilde{M}_{u,v}=\int\int_{T}\left(\int\int_{T}f''(B^{g,\gamma}_{u,v})1_{[0,s]}(u)1_{[0,t]}(v)1_{[0,u]}(u')1_{[0,v]}(v')B^{g,\gamma}(du',dv)\right)B^{g,\gamma}(du,dv')$$

and $d_{u}W^{\alpha,\beta}_{u,v}$ denotes the Skorohod differential of the one-parameter Gaussian process $u\to B^{g,\gamma}_{u,v}$.

We leave the statement of the corresponding Tanaka formula to the reader, who will only need to apply the substitutions given immediately preceding the above theorem to the statement of Proposition 2.

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