

# Large deviations for rough paths of the fractional Brownian motion

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**Abstract:** Starting from the construction of a geometric rough path associated with a fractional Brownian motion with Hurst parameter  $H \in ]\frac{1}{4}, \frac{1}{2}[$  given by Coutin and Qian in [4], we prove a large deviation principle in the space of geometric rough paths, extending classical results on Gaussian processes. As a by-product, geometric rough paths associated to elements of the reproducing kernel Hilbert space of the fractional Brownian motion are obtained and an explicit integral representation is given.

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# 1 Introduction

In the seminal paper [14], Lyons developed a sophisticated mathematical theory to analyse dynamical systems with an external rough force acting as a control and influencing their evolution. One of the key ideas is to keep the non-commutative structure of controls on small time steps. Rough controls are constructed as elements of direct sums of tensor spaces endowed with a topology associated with the  $p$ -variation distance. Dynamical systems are proved to be continuous functionals of their rough path controls with respect to this topology. This result is called the *universal limit theorem*.

Stochastic modeling deals basically with rough path controls. Indeed, the ground-breaking Itô's theory on stochastic differential equations is based on Brownian motion, which has almost surely nowhere differentiable sample paths but only  $\alpha$ -Hölder continuous ones, with  $\alpha \in ]0, \frac{1}{2}[$ . Note that the solution of a multidimensional stochastic Itô's differential equation is not a continuous functional of the driving Brownian motion. From Lyons perspective, the rough path character of Brownian motion is caught by increments of both, its trajectories and those of the Lévy area process. His approach provides a kind of pathwise calculus well-suited for system control in a stochastic context. We refer the reader to [15] and [13], where the basic ingredients of the theory are presented.

Itô's theory has been extensively developed in many different directions, including finite and infinite dimensional settings. Recently, increasing attention is being devoted to a particular stochastic control rougher than the Brownian motion: the fractional Brownian motion with Hurst parameter  $H \in ]0, \frac{1}{2}[$ . Unlike the classical Brownian process ( $H = \frac{1}{2}$ ), the fractional Brownian motion does not have independent increments and possesses long-range memory. Many problems in traffic networks, hydrology and economics, just to mention a few examples, share these properties and therefore can be realistically analysed including this process in their mathematical formulation. In [16] a large survey on fractional Brownian motion is given. Some of the recent developments concerning fractional Brownian motion are employed in this paper (see for instance [1], [4], [2], [5]). These references contain an exhaustive list of contributors to the subject and are suggested to those who would like to have a broad picture on the subject.

In this article, we are interested in the rough path associated with a fractional Brownian motion with Hurst parameter  $H \in ]\frac{1}{4}, \frac{1}{2}[$ , constructed in [4]. The main goal has been to establish a large deviation principle. For  $H = \frac{1}{2}$ , this question has been addressed in [12] and the possibility of the extension given in our work is mentioned. However, we believe that it is not a straightforward one and gives rise to interesting mathematical issues

which need new ideas to be solved satisfactory. For values of  $H$  in  $]\frac{1}{2}, 1[$ , the problem has an almost obvious answer -see the remark following the proof of Proposition 4.

In order to give a more detailed description of the results in their context, some basic notions on rough paths analysis and some notation should be set up.

Let  $T > 0$  and  $\mathbb{B}$  be a Banach space. For  $p \geq 1$ , the  $p$ -variation norm of a function  $x : [0, T] \rightarrow \mathbb{B}$  is defined by

$$\|x\|_p = \left( \sup_{\mathcal{P}} \sum_l |x_{t_l} - x_{t_{l-1}}|_{\mathbb{B}}^p \right)^{\frac{1}{p}},$$

where the supremum runs over all finite partitions  $\mathcal{P}$  of  $[0, T]$ . In the sequel we shall take  $T = 1$  and consider  $\mathbb{B} = \mathbb{R}^d$ .

A continuous map  $X$  defined on the simplex  $\Delta = \{(s, t) : 0 \leq s \leq t \leq 1\}$ , taking values on the truncated tensor algebra

$$T^{[p]}(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{R}^d)^{\otimes [p]}$$

is called a *rough path* in  $T^{[p]}(\mathbb{R}^d)$  of roughness  $p$ , if  $X_{s,t} = (1, X_{s,t}^1, \dots, X_{s,t}^{[p]})$ ,  $(s, t) \in \Delta$ , satisfies the properties:

- (a) Finite  $p$ -variation:  $\max_{1 \leq j \leq [p]} \left( \sup_{\mathcal{P}} \sum_l |X_{t_{l-1}, t_l}^j|_{\mathbb{R}^d}^{\frac{p}{j}} \right)^{\frac{j}{p}} < \infty$ .
- (b) Multiplicative property:  $X_{s,t} = X_{s,u} \otimes X_{u,t}$ , for any  $(s, u), (u, t) \in \Delta$ .

The set of rough paths in  $T^{[p]}(\mathbb{R}^d)$  is a metric space with the  $p$ -variation distance

$$d_p(X, Y) = \max_{1 \leq j \leq [p]} \left( \sup_{\mathcal{P}} \sum_l |X_{t_{l-1}, t_l}^j - Y_{t_{l-1}, t_l}^j|_{\mathbb{R}^d}^{\frac{p}{j}} \right)^{\frac{j}{p}}. \quad (1)$$

Assume that the function  $x$  has finite total variation. For any  $j = 1, \dots, [p]$ ,  $(s, t) \in \Delta$ , consider the  $j$ -th iterated integral

$$X_{s,t}^j = \int \dots \int_{s < t_1 < \dots < t_j < t} dx_{t_1} \otimes \dots \otimes dx_{t_j}. \quad (2)$$

It is easy to check that  $X_{s,t} = (1, X_{s,t}^1, \dots, X_{s,t}^{[p]})$  defined in this way is a rough path. We shall refer to this class of objects as *smooth rough paths* lying above  $x$ .

The space of *geometric rough paths* with roughness  $p$  is the closure of the set of smooth rough paths with respect to the  $p$ -variation metric. An important class in stochastic analysis of geometric rough paths are those obtained

from smooth rough paths based on linear interpolations of  $x$ . They shall be denoted by  $\mathcal{D}_p(\mathbb{R}^d)$ . Indeed, linear interpolations of interesting examples like Brownian motion,  $\mathbb{B}$ -valued Wiener process, free Brownian motion and fractional Brownian motion have been successfully used to define the corresponding geometric rough path (see [15], [11], [3], [4], respectively).

In this paper, we consider a  $d$ -dimensional fractional Brownian motion  $W^H = (W_t^H, t \in [0, 1])$  with Hurst parameter  $H \in ]0, 1[$ . Its reproducing kernel Hilbert space, denoted by  $\mathcal{H}^H$ , consists of functions  $h : [0, 1] \rightarrow \mathbb{R}^d$  that can be represented as

$$h(t) = \int_0^t K^H(t, s) \dot{h}(s) ds, \quad (3)$$

where  $K^H(t, s)$  is the kernel defined by

$$\begin{aligned} K^H(t, s) &= c_H (t - s)^{H - \frac{1}{2}} \\ &+ c_H \left( \frac{1}{2} - H \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - H} \right) du, \end{aligned} \quad (4)$$

for  $c_H > 0$ ,  $0 < s < t \leq 1$ , and  $\dot{h} \in L^2([0, 1])$ . The scalar product in  $\mathcal{H}^H$  is given by

$$\langle h_1, h_2 \rangle_{\mathcal{H}^H} = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2[0, 1]}$$

(see [5], Theorem 3.3).

For  $0 < s < t$

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{\frac{1}{2} - H} (t - s)^{H - \frac{3}{2}}. \quad (5)$$

Note that for  $H \in ]0, \frac{1}{2}[$ ,  $|K^H|(dt, s) = -\frac{\partial K^H}{\partial t}(t, s) 1_{]s, 1[}(t) dt$ .

Let  $\mathbb{E} = \mathcal{C}_0([0, 1]; \mathbb{R}^d)$ , endowed with the topology of the supremum norm and let  $P^H$  be the law of  $W^H$  on  $\mathbb{E}$ . The triple  $(\mathbb{E}, \mathcal{H}^H, P^H)$  is an abstract Wiener space. We shall denote by  $i^H$  the continuous dense embedding of  $\mathcal{H}^H$  into  $\mathbb{E}$ .

A classical result of the theory of Gaussian processes (see for instance [8], Theorem 3.4.12) establishes that the family  $(\epsilon P^H, \epsilon > 0)$  of Gaussian probabilities satisfies a large deviation principle on  $\mathbb{E}$  with good rate function

$$\Lambda^H(x) = \begin{cases} \frac{1}{2} \| (i^H)^{-1}(x) \|_{\mathcal{H}^H}^2 & \text{if } x \in i^H(\mathcal{H}^H), \\ +\infty & \text{otherwise.} \end{cases} \quad (6)$$

Along this article we deal with values of  $H$  in  $]\frac{1}{4}, \frac{1}{2}[$ . For the sake of simplicity, we shall skip any reference to the parameter  $H$  in the sequel and write  $W$  instead of  $W^H$ ,  $\mathcal{H}$  instead of  $\mathcal{H}^H$ , etc.

For any  $m \in \mathbb{N}$ , we consider the  $m$ -th dyadic grid ( $t_l^m = l2^{-m}, l = 0, 1, \dots, 2^m$ ) and set  $\Delta_l^m W = W_{t_l^m} - W_{t_{l-1}^m}$ , for any  $l = 1, \dots, 2^m$ .

Denote by  $W(m) = (W(m)_t, t \in [0, 1])$  the process obtained by linear interpolation of  $W$  on the  $m$ -th dyadic grid. That is,  $W(m)_0 = 0$  and for  $t \in ]t_{l-1}^m, t_l^m]$ ,

$$W(m)_t = W_{t_{l-1}^m} + 2^m(t - t_{l-1}^m) \Delta_l^m W. \quad (7)$$

Let  $p \in ]1, 4[$  be such that  $Hp > 1$ . In [4], a geometric rough path with roughness  $p$ , lying above  $W$  is obtained as a limit in the  $p$ -variation distance (1) of the sequence of smooth rough paths  $F(W(m)) = (1, W(m)^1, W(m)^2, W(m)^3)$  defined as in (2). We denote this object by  $F(W)$ . By its very construction,  $F(W) \in \mathcal{D}_p(\mathbb{R}^d)$ .

As has been mentioned before, our purpose is to establish a large deviation principle on  $\mathcal{D}_p(\mathbb{R}^d)$  for the family of probability laws of  $(F(\epsilon W), \epsilon \in (0, 1))$ , extending the classical Schilder result for Gaussian processes. By means of the universal limit theorem of [14], the result can be transferred to stochastic differential equations driven by fractional Brownian motion.

The next section 2 is devoted to the proof of the main result. We follow the same strategy as in [12]. That is, since the smooth rough paths based on linear interpolations of the process  $W$  are easily seen to satisfy a large deviation principle, we only need to prove that they are *exponentially good* approximations of  $W$ . In comparison with [12], there are essentially two new difficulties coming up. Firstly, time increments of fractional Brownian motion are not independent and secondly, we need to deal with third order geometric rough paths, making arguments a bit more involved. The main tools to be used are the hypercontractivity inequality for Gaussian chaos (see [10]) and a collection of covariance type estimates for  $W$  proved in [4]. As a by-product, we prove the existence of a geometric rough path associated with each element  $h$  in the reproducing kernel Hilbert space  $\mathcal{H}$ . Section three is entirely devoted to give a precise description of this geometric rough path in terms of indefinite multiple integrals. The results might be understood as deterministic versions of those given in [1] for stochastic integrals with respect to Gaussian Volterra processes (see also [6]). In our case, integrands and integrators are of Volterra type, because of the representation (3). The interest of these results goes beyond the framework of this work; they shall be useful in the characterization of the topological support of the law of the rough path associated with the fractional Brownian motion.

As is being usual, we denote throughout the proofs different constants by the same letter.

## 2 The large deviation principle

We want to prove the following.

**Theorem 1** *Let  $H \in ]\frac{1}{4}, \frac{1}{2}[$ ,  $p \in ]1, 4[$  be such that  $Hp > 1$ . The family of probability laws of  $(F(\epsilon W), \epsilon \in (0, 1))$  satisfies a large deviation principle on  $\mathcal{D}_p(\mathbb{R}^d)$  with the good rate function defined for  $X \in \mathcal{D}_p(\mathbb{R}^d)$  by:*

$$\mathcal{I}(X) = \begin{cases} \frac{1}{2} \|i^{-1}(X_{0,\cdot}^1)\|_{\mathcal{H}}^2 & \text{if } X_{0,\cdot}^1 \in i(\mathcal{H}), \\ +\infty & \text{otherwise.} \end{cases} \quad (8)$$

Let us start by setting the method of the proof, that we borrow from [12] and fix the notations to be used in the sequel.

Let  $Z(m) = (W_{t_l^m}, 1 \leq l \leq 2^m)$ . Clearly,  $Z(m) = \Phi_m \circ W$ , with  $\Phi_m : \mathbb{E} \rightarrow (\mathbb{R}^d)^{2^m}$  a continuous map. The explicit form of the smooth rough path lying above  $W(m)$  shows that there exists a continuous map  $\Psi_m : (\mathbb{R}^d)^{2^m} \rightarrow \mathcal{D}_p(\mathbb{R}^d)$  such that  $F(W(m)) = (\Psi_m \circ \Phi_m)(W)$ . Consequently, the contraction principle implies that for any  $m$  the family of probability laws of  $(F(\epsilon W(m)), \epsilon \in (0, 1))$  satisfies a large deviation principle on  $\mathcal{D}_p(\mathbb{R}^d)$  with the good rate function

$$\mathcal{I}_m(X) = \inf\{\Lambda(x) : x \in \mathbb{E}, (\Psi_m \circ \Phi_m)(x) = X\}, \quad (9)$$

$X \in \mathcal{D}_p(\mathbb{R}^d)$ .

We then transfer the large deviation principle from  $F(\epsilon W(m))$  to  $F(\epsilon W)$ . At first we shall prove that for any  $\delta > 0$ ,

$$\lim_{m \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \epsilon^2 \log P(d_p(F(\epsilon W(m)), F(\epsilon W)) > \delta) = -\infty. \quad (10)$$

For any  $h \in \mathcal{H}$ , let  $h(m)$  denote the smooth function obtained by linear interpolation of  $h$  on the  $m$ -th dyadic grid and let  $F(h(m))$  be the corresponding smooth rough path.

We will prove that for every  $\alpha > 0$ ,

$$\lim_{m, m' \rightarrow \infty} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} d_p(F(h(m)), F(h(m'))) = 0. \quad (11)$$

This result gives in particular the existence of a geometric rough path  $F(h)$  in  $\mathcal{D}_p(\mathbb{R}^d)$  obtained as the limit in the  $d_p$ -variation distance of  $F(h(m))$ . In the last part of the article we shall identify  $F(h)$  as a triple of integrals.

By means of an extension of the contraction principle (see [7], Theorem 4.2.23), (10), (11) provide a proof of Theorem 1.

Let us introduce some technicalities to deal with the  $p$ -variation distance  $d_p$ .

If  $X, Y$  are rough paths of degree  $[p]$ , we set for  $j = 1, \dots, [p]$ ,  $\gamma > 0$ ,

$$D_{j,p}(X, Y) = \left( \sum_{n=1}^{\infty} n^\gamma \sum_{l=1}^{2^n} |X_{t_{l-1}^n, t_l^n}^j - Y_{t_{l-1}^n, t_l^n}^j|^{\frac{p}{j}} \right)^{\frac{j}{p}}$$

and  $D_{j,p}(X) = D_{j,p}(X, 0)$ .

Owing to results proved in [11] and [4] (see also [15]), for any  $p \in ]3, 4[$ ,  $\gamma > p - 1$ ,

$$\begin{aligned} d_p(X, Y) \leq C \max & \left( D_{1,p}(X, Y), D_{1,p}(X, Y) [D_{1,p}(X) + D_{1,p}(Y)], D_{2,p}(X, Y), \right. \\ & D_{2,p}(X, Y) [D_{1,p}(X) + D_{1,p}(Y)], D_{1,p}(X, Y) [D_{2,p}(X) + D_{2,p}(Y)], \\ & \left. D_{1,p}(X, Y) [D_{1,p}(X)^2 + D_{1,p}(Y)^2], D_{3,p}(X, Y) \right). \end{aligned} \quad (12)$$

Therefore, similar arguments as in [12], pp. 273-274 show that (10) follows from the following statement.

**Proposition 2** *Let  $p \in ]1, 4[$  be such that  $pH > 1$ . Then,*

(a) *For any  $j = 1, 2, 3$ , there exists a sequence  $c_j(m)$  converging to zero as  $m$  tends to infinity such that for every  $q > p$ ,*

$$\left( E (D_{j,p}(W(m), W)^q) \right)^{\frac{1}{q}} \leq c_j(m) q^{\frac{j}{2}}. \quad (13)$$

(b) *For any  $j = 1, 2$ , there exists a constant  $c_j$  such that for every  $q > p$ ,*

$$\sup_{m \in \mathbb{N}} \left( E (D_{j,p}(W(m))^q) \right)^{\frac{1}{q}} \leq c_j q^{\frac{j}{2}}. \quad (14)$$

**Proof:** We shall denote by  $g$  a standard normal random variable and observe that, as a consequence of the hypercontractivity inequality (see e.g. [10], page 65),  $(E|g|^q)^{\frac{1}{q}} \leq (q-1)^{\frac{1}{2}}$ , for any  $q \in ]2, \infty[$ . Along the proof, for any  $n \geq m$ ,  $l = 1, \dots, 2^n$ , we denote by  $k := k(n, m, l)$  the unique integer in  $\{1, 2, \dots, 2^m\}$  such that

$$t_{k-1}^m \leq t_{l-1}^n < t_l^n < t_k^m. \quad (15)$$

*First order terms.* Let  $j = 1$ . From the definition of  $W(m)$ <sup>1</sup> it follows easily,

$$D_{1,p}(W(m), W) = \left( \sum_{n=m+1}^{\infty} n^\gamma \sum_{l=1}^{2^n} |2^{m-n} \Delta_k^m W - \Delta_l^n W|^p \right)^{\frac{1}{p}}.$$

As in [12], for  $q > p$  and  $m \geq 0$  set  $A(m, q) = \left( \sum_{n=m+1}^{\infty} 2^n \left( \frac{n^\gamma}{a_n} \right)^{\frac{q}{q-p}} \right)^{\frac{q-p}{p}}$ , for some sequence of real positive numbers  $(a_n, n \geq 1)$  to be chosen later.

Hölder's inequality yields

$$\begin{aligned} E\left(D_{1,p}(W(m), W)^q\right) &\leq A(m, q) \sum_{n=m+1}^{\infty} a_n^{\frac{q}{p}} \sum_{l=1}^{2^n} E\left(|2^{m-n} \Delta_k^m W - \Delta_l^n W|^q\right) \\ &\leq A(m, q) (2d)^q q^{\frac{q}{2}} \sum_{n=m+1}^{\infty} a_n^{\frac{q}{p}} 2^n \left(2^{-nq} 2^{mq(1-H)} + 2^{-nqH}\right) \\ &\leq A(m, q) (2d)^q q^{\frac{q}{2}} \sum_{n=m+1}^{\infty} a_n^{\frac{q}{p}} 2^{n(1-qH)}. \end{aligned} \quad (16)$$

Set  $a_n = 2^{np(H - \frac{1}{q} - \epsilon)}$  for some  $\epsilon \in ]0, \frac{1}{2}(H - \frac{1}{p})[$ ; then the series  $\sum_n a_n^{\frac{q}{p}} 2^{n(1-Hq)}$  converges. Fix  $\delta > 0$  such that  $n^\gamma \leq c 2^{n\delta p}$  for some  $c > 0$  and  $\epsilon + \delta \in ]0, \frac{1}{2}(H - \frac{1}{p})[$ . Then,

$$A(m, q)^{\frac{1}{q}} \leq c^{\frac{1}{p}} 2^{-m(H - \frac{1}{p} - \epsilon - \delta)}.$$

Consequently,  $\sup_{q>p} A(m, q)^{\frac{1}{q}}$  tends to zero as  $m$  tends to infinity. By virtue of (16), the upper bound (13) for  $j = 1$  holds true.

The proof of (14) for  $j = 1$  is similar. Indeed, proceeding as for the proof of (16), we have

$$E\left(D_{1,p}(W)^q\right) \leq A(0, q) (2d)^q q^{\frac{q}{2}} \sum_{n=1}^{\infty} a_n^{\frac{q}{p}} 2^{n(1-qH)}. \quad (17)$$

Since  $\sup_{q>p} A(0, q) < \infty$ , the inequalities (16), (17) yield (14).

*Second order terms.* Let  $j = 2$ . For  $l = 1, \dots, 2^n$  set

$$T_2(n, m, l) = W(m+1)_{t_{l-1}^n, t_l^n}^2 - W(m)_{t_{l-1}^n, t_l^n}^2.$$

Assume first  $n < m$ . Quoting equation (20) in [4],

$$T_2(n, m, l) = \frac{1}{2} \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}l} \left( \Delta_{2r-1}^{m+1} W \otimes \Delta_{2r}^{m+1} W - \Delta_{2r}^{m+1} W \otimes \Delta_{2r-1}^{m+1} W \right).$$

Clearly,  $T_2(n, m, l) = 0$  for  $d = 1$  and for any  $d \geq 2$ , all the diagonal components  $T_2(n, m, l)^{i,i}$  vanish. Hence, we may assume that  $d \geq 2$  and consider only  $(i, j)$  components with  $i \neq j$ . Under these premises, any couple of random variables  $\Delta_k^{m+1} W^i, \Delta_h^{m+1} W^j$  are independent.



Owing to the hypercontractivity inequality,

$$\left(E \left|T_2(n, m, l)^{i,j}\right|^q\right)^{\frac{1}{q}} \leq Cq \left(E \left|T_2(n, m, l)^{i,j}\right|^2\right)^{\frac{1}{2}}.$$

Clearly,

$$E \left|T_2(n, m, l)^{i,j}\right|^2 \leq C \left(T_{21}(n, m, l)^{i,j} + T_{22}(n, m, l)^{i,j}\right),$$

where

$$\begin{aligned} T_{21}(n, m, l)^{i,j} &= \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}l} E \left| \Delta_{2^{r-1}}^{m+1} W_i \Delta_{2^r}^{m+1} W_j - \Delta_{2^{r-1}}^{m+1} W_j \Delta_{2^r}^{m+1} W_i \right|^2 \\ &\leq C \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}l} E \left| \Delta_{2^{r-1}}^{m+1} W_i \right|^2 E \left| \Delta_{2^r}^{m+1} W_j \right|^2 \\ &\leq C 2^{-n} 2^{-4m(H-\frac{1}{4})}. \end{aligned} \quad (18)$$

Lemma 12 in [4] yields

$$\begin{aligned} T_{22}(n, m, l)^{i,j} &\leq C \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}l} \left( \sum_{\bar{r}=r+1}^{\infty} (\bar{r} - r)^{4H-5} \right) 2^{-4(m+1)H} \\ &\leq C 2^{-n} 2^{-4m(H-\frac{1}{4})}. \end{aligned} \quad (19)$$

Consequently,

$$\left(E \left|T_2(n, m, l)^{i,j}\right|^q\right)^{\frac{1}{q}} \leq Cq 2^{-\frac{n}{2}} 2^{-2m(H-\frac{1}{4})}. \quad (20)$$

This inequality holds also true for  $n = m$ . Indeed, using for instance the identities (5) and (6) in [12] for  $m = n + 1$  and  $m = n$ , respectively, we obtain

$$T_2(n, n, l) = \frac{1}{2} \left( \Delta_{2^{l-1}}^{n+1} W \otimes \Delta_{2^l}^{n+1} W - \Delta_{2^l}^{n+1} W \otimes \Delta_{2^{l-1}}^{n+1} W \right),$$

and therefore,

$$\left(E \left|T_2(n, n, l)^{i,j}\right|^q\right)^{\frac{1}{q}} \leq Cq 2^{-2nH}.$$

Fix  $M > m$ . The above inequality (20) and Minkowski's inequality imply

$$\begin{aligned} \left(E \left|W(M)_{t_{l-1}^m, t_l^m}^2 - W(m)_{t_{l-1}^m, t_l^m}^2\right|^q\right)^{\frac{1}{q}} &\leq Cq 2^{-\frac{n}{2}} \sum_{N=m}^{M-1} 2^{-2N(H-\frac{1}{4})} \\ &\leq Cq 2^{-\frac{n}{2}} 2^{-2m(H-\frac{1}{4})}, \end{aligned} \quad (21)$$

where  $C$  is a constant depending only on  $H, p$  and  $d$ .

By the construction of the rough path lying above  $W$ , a.s.,

$$\lim_{M \rightarrow \infty} W(M)_{t_{i-1}^n, t_i^n}^2 = W_{t_{i-1}^n, t_i^n}^2.$$

Therefore, Fatou's lemma and (21) yield for  $n \leq m$

$$\left( E \left| W_{t_{i-1}^n, t_i^n}^2 - W(m)_{t_{i-1}^n, t_i^n}^2 \right|^q \right)^{\frac{1}{q}} \leq Cq2^{-\frac{n}{2}}2^{-2m(H-\frac{1}{4})}. \quad (22)$$

Let  $m \leq n$ ; in this case,

$$W(m)_{t_{i-1}^n, t_i^n}^2 = 2^{2(m-n)-1}(\Delta_k^m W)^{\otimes 2},$$

where  $k = k(n, m, l)$  satisfies (15) (see [4], equation (17)). By the hypercontractivity property,

$$\left( E \left| W(m)_{t_{i-1}^n, t_i^n}^2 \right|^q \right)^{\frac{1}{q}} \leq Cq2^{-2n}2^{-2m(H-1)}. \quad (23)$$

The previous estimate (22) for  $n = m$  together with Minkowski's inequality, imply

$$\left( E \left| W_{t_{i-1}^n, t_i^n}^2 \right|^q \right)^{\frac{1}{q}} \leq Cq2^{-2nH}. \quad (24)$$

With (23) and (24) we obtain for  $m \leq n$ ,

$$\left( E \left| W_{t_{i-1}^n, t_i^n}^2 - W(m)_{t_{i-1}^n, t_i^n}^2 \right|^q \right)^{\frac{1}{q}} \leq Cq2^{-2nH}. \quad (25)$$

We now proceed in an analogue way as for  $j = 1$ . For  $q > \frac{p}{2}$ , set  $A_2(q) = \left( \sum_{n=1}^{\infty} 2^n \left( \frac{n^\gamma}{a_n} \right)^{\frac{2q}{2q-p}} \right)^{\frac{2q-p}{p}}$ , for some positive real sequence  $(a_n, n \geq 1)$ . By Hölder's inequality,

$$E(D_{2,p}(W(m), W)^q) \leq A_2(q) \sum_{n=1}^{\infty} a_n^{\frac{2q}{p}} \sum_{l=1}^{2^n} E \left| W(m)_{t_{l-1}^n, t_l^n}^2 - W_{t_{l-1}^n, t_l^n}^2 \right|^q. \quad (26)$$

From (22) and (25), it follows that

$$\begin{aligned} E \left( D_{2,p}(W(m), W)^q \right) &\leq CA_2(q)q^q \left[ \sum_{n=1}^m a_n^{\frac{2q}{p}} 2^{-n(\frac{q}{2}-1)} 2^{-2mq(H-\frac{1}{4})} \right. \\ &\quad \left. + \sum_{n=m+1}^{\infty} a_n^{\frac{2q}{p}} 2^{-n(2qH-1)} \right]. \end{aligned}$$

Notice that, for any  $\eta \in ]0, 2H - \frac{1}{2}[$ ,

$$\sum_{n=1}^m a_n^{\frac{2q}{p}} 2^{-n(\frac{q}{2}-1)} 2^{-2mq(H-\frac{1}{4})} \leq 2^{-mq\eta} \sum_{n=1}^{\infty} a_n^{\frac{2q}{p}} 2^{-n[q(2H-\eta)-1]}. \quad (27)$$

Let  $a_n = 2^{-np(\epsilon-H+\frac{\eta}{2}+\frac{1}{2q})}$ , with  $\epsilon > 0$ . Then the series  $\sum_n a_n^{\frac{2q}{p}} 2^{-n[q(2H-\eta)-1]}$  converges. Moreover, this choice of  $a_n$  yields

$$A_2(q) = \left( \sum_{n=1}^{\infty} n^{\frac{2\gamma q}{2q-p}} 2^{-\frac{npq}{2q-p}(-2\epsilon+2H-\eta-\frac{2}{p})} \right)^{\frac{2q-p}{p}},$$

$$\sum_{n=m+1}^{\infty} a_n^{\frac{2q}{p}} 2^{-n(2qH-1)} = \sum_{n=m+1}^{\infty} 2^{-nq(\eta+2\epsilon)}.$$

Let  $\eta, \epsilon$  and  $\delta$  be positive reals such that  $\delta + \epsilon + \frac{\eta}{2} < H - \frac{1}{p}$ , and  $n^\gamma \leq C2^{np\delta}$ , for some  $C > 0$ . Then  $\sup_{q>\frac{p}{2}} (A_2(q))^{\frac{1}{q}} < \infty$  and consequently,

$$\left( E(D_{2,p}(W(m), W)^q) \right)^{\frac{1}{q}} \leq Cq2^{-m\eta}, \quad (28)$$

proving (13) for  $j = 2$ .

By a similar approach, using the estimate (24), we can prove that

$$\left( E(D_{2,p}(W)^q) \right)^{\frac{1}{q}} \leq Cq.$$

Thus, (14) for  $j = 2$  holds true.

*Third order terms.* Finally, let us prove (13) for  $j = 3$ . Assume first  $n \leq m$ ; then for any  $l = 1, \dots, 2^n$ ,

$$E \left| W(m+1)_{t_{l-1}^n, t_l^n}^3 - W(m)_{t_{l-1}^n, t_l^n}^3 \right|^2 \leq C2^{-n(1+2H)} 2^{-m(4H-1)}. \quad (29)$$

Indeed, for  $n < m$ , the inequality is proved in [4], pg. 128. Let  $n = m$ ; quoting [4] pg. 119, for any  $n \geq 1$ , we write

$$W(n+1)_{t_{l-1}^n, t_l^n}^3 = \sum_{k=2l-1}^{2l} \left( W(n+1)_{t_{k-1}^{n+1}, t_k^{n+1}}^3 \right. \\ \left. + W(n+1)_{t_{l-1}^n, t_{k-1}^{n+1}}^1 \otimes W(n+1)_{t_{k-1}^{n+1}, t_k^{n+1}}^2 \right. \\ \left. + W(n+1)_{t_{l-1}^n, t_{k-1}^{n+1}}^2 \otimes W(n+1)_{t_{k-1}^{n+1}, t_k^{n+1}}^1 \right). \quad (30)$$

Fix  $k \in \{2l-1, 2l\}$ ; it is easy to check that for any  $q \in [2, \infty)$ ,

$$\left( E \left| W(n+1)_{t_{l-1}^n, t_{k-1}^{n+1}}^1 \right|^q \right)^{\frac{1}{q}} + \left( E \left| W(n+1)_{t_{k-1}^{n+1}, t_k^{n+1}}^1 \right|^q \right)^{\frac{1}{q}} \leq Cq^{\frac{1}{2}} 2^{-(n+1)H}. \quad (31)$$

Applying (23) we obtain

$$\left(E \left| W(n+1)_{t_{k-1}^{n+1}, t_k^{n+1}}^2 \right|^q\right)^{\frac{1}{q}} + \left(E \left| W(n+1)_{t_{l-1}^{n+1}, t_k^{n+1}}^2 \right|^q\right)^{\frac{1}{q}} \leq Cq2^{-(2n+2)H}. \quad (32)$$

Moreover, for any  $m \leq n$ ,

$$W(m)_{t_{l-1}^n, t_l^n}^3 = \frac{2^{3(m-n)}}{3!} (\Delta_k^m W)^{\otimes 3}, \quad (33)$$

with  $k = k(n, m, l)$  satisfying (15). Since  $W(m)_{t_{l-1}^n, t_l^n}^3$  belongs to the third order Gaussian chaos, the hypercontractivity property yields for  $m \leq n$ ,

$$\left(E \left| W(m)_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-3n} 2^{-3m(H-1)}. \quad (34)$$

From (30) - (32) and (34), we obtain

$$E \left| W(n+1)_{t_{l-1}^n, t_l^n}^3 \right|^2 \leq C2^{-6nH}. \quad (35)$$

This upper bound, together with (34) for  $m = n$  and  $q = 2$ , imply the validity of (29) for  $n = m$ .

By virtue of the hypercontractivity property and (29) we deduce for  $n \leq m$ ,

$$\left(E \left| W(m+1)_{t_{l-1}^n, t_l^n}^3 - W(m)_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-n(\frac{1}{2}+H)} 2^{-m(2H-\frac{1}{2})}.$$

Hence, Minkowski's inequality yields

$$\left(E \left| W(M)_{t_{l-1}^n, t_l^n}^3 - W(m)_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-n(\frac{1}{2}+H)} 2^{-m(2H-\frac{1}{2})}$$

for any  $M > m \geq n$ .

We observe that, a.s.  $\lim_{M \rightarrow \infty} W(M)_{t_{l-1}^n, t_l^n}^3 = W_{t_{l-1}^n, t_l^n}^3$ . Therefore, Fatou's Lemma yields for  $m \geq n$ ,

$$\left(E \left| W_{t_{l-1}^n, t_l^n}^3 - W(m)_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-n(\frac{1}{2}+H)} 2^{-m(2H-\frac{1}{2})}. \quad (36)$$

Suppose  $m \leq n$ . Applying the previous estimate (36) and (34) with  $m = n$ , we obtain

$$\left(E \left| W_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-3nH}.$$

Therefore, using again (34) we deduce for  $m \leq n$ ,

$$\left(E \left| W_{t_{l-1}^n, t_l^n}^3 - W(m)_{t_{l-1}^n, t_l^n}^3 \right|^q\right)^{\frac{1}{q}} \leq Cq^{\frac{3}{2}} 2^{-3nH}. \quad (37)$$

For  $q > \frac{p}{3}$ , let  $A_3(q) = \left( \sum_{n=1}^{\infty} 2^n \left( \frac{n^\gamma}{a_n} \right)^{\frac{3q}{3q-p}} \right)^{\frac{3q-p}{p}}$ , where  $a_n$  is a sequence of positive numbers to be determined later. Hölder's inequality yields

$$E\left(D_{3,p}(W(m), W)^q\right) \leq A_3(q) \sum_{n=1}^{\infty} a_n^{\frac{3q}{p}} \sum_{l=1}^{2^n} E \left| W(m)_{t_{l-1}^n, t_l^n}^3 - W_{t_{l-1}^n, t_l^n}^3 \right|^q.$$

By means of (36), (37) we obtain,

$$\begin{aligned} E\left(D_{3,p}(W(m), W)^q\right) &\leq A_3(q) q^{\frac{3q}{2}} \left( \sum_{n=1}^m a_n^{\frac{3q}{p}} 2^{-nq(\frac{1}{2}+H-\frac{1}{q})-mq(2H-\frac{1}{2})} \right. \\ &\quad \left. + \sum_{n=m+1}^{\infty} a_n^{\frac{3q}{p}} 2^{-n(3qH-1)} \right). \end{aligned}$$

Let  $\eta \in ]0, 2H - \frac{1}{2}[$ ; clearly,

$$\sum_{n=1}^m a_n^{\frac{3q}{p}} 2^{-nq(\frac{1}{2}+H-\frac{1}{q})-mq(2H-\frac{1}{2})} \leq 2^{-mq\eta} \sum_{n=1}^m a_n^{\frac{3q}{p}} 2^{-nq(3H-\frac{1}{q}-\eta)}.$$

Set  $a_n = 2^{-np(\frac{\epsilon}{3}-H+\frac{\eta}{3}+\frac{1}{3q})}$ , with  $\epsilon > 0$ . Then the series  $\sum_{n=1}^{\infty} a_n^{\frac{3q}{p}} 2^{-nq(3H-\frac{1}{q}-\eta)}$  converges. Furthermore,

$$\begin{aligned} A_3(q) &= \left( \sum_{n=1}^{\infty} n^{\frac{3\gamma q}{3q-p}} 2^{-\frac{npq}{3q-p}(-\epsilon+3H-\eta-\frac{3}{p})} \right)^{\frac{3q-p}{p}}, \\ \sum_{n=m+1}^{\infty} a_n^{\frac{3q}{p}} 2^{-n(3qH-1)} &= \sum_{n=m+1}^{\infty} 2^{-nq(\epsilon+\eta)}. \end{aligned}$$

Let  $\eta > 0, \epsilon > 0$  and  $\delta > 0$  be such that  $3\delta + \epsilon + \eta < 3H - \frac{3}{p}$  and  $n^\gamma < C2^{np\delta}$ , for some  $C > 0$ . Then  $\sup_{q>\frac{p}{3}} (A_3(q))^{\frac{1}{q}} < \infty$ . Thus,

$$\left( E(D_{3p}(W(m), W)^q) \right)^{\frac{1}{q}} \leq Cq2^{-m\eta},$$

proving (13) for  $j = 3$ . This concludes the proof of the Proposition.  $\square$

In the sequel, we make the convention  $K(t, s) = 0$  if  $s \geq t$ , and therefore write

$$h(t) = \int_0^1 K(t, s) \dot{h}(s) ds,$$

for any  $h \in \mathcal{H}$ . We denote by  $\|\cdot\|_2$  the usual Hilbert norm in  $L^2([0, 1])$ .

**Lemma 3** *Let  $h \in \mathcal{H}$  and  $t, t' \in [0, 1]$ . Then*

$$|h(t) - h(t')| \leq \|\dot{h}\|_2 |t - t'|^H. \quad (38)$$

*In particular, for any  $\alpha > 0$ ,*

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} |h(t) - h(t')| \leq \alpha |t - t'|^H. \quad (39)$$

**Proof:** With the above convention on the kernel  $K$  and by virtue of Schwarz's inequality, we have

$$\begin{aligned} |h(t) - h(t')|^2 &= \left| \int_0^1 (K(t, s) - K(t', s)) \dot{h}(s) ds \right|^2 \\ &\leq \|\dot{h}\|_2^2 \int_0^1 (K(t, s) - K(t', s))^2 ds \\ &= \|\dot{h}\|_2^2 E |W_t - W_{t'}|^2 = \|\dot{h}\|_2^2 |t - t'|^{2H}. \quad \square \end{aligned}$$

In the remaining part of the section,  $h$  shall denote a fixed element in  $\mathcal{H}$  and  $h(m)$ ,  $m \geq 1$ , the function obtained by linear interpolation of  $h$  on the  $m$ -th dyadic grid ( $t_l^m = l2^{-m}$ ,  $l = 0, 1, \dots, 2^m$ ). That is,  $h(m)_0 = 0$  and for  $t \in ]t_{l-1}^m, t_l^m]$ ,

$$h(m)_t = h(t_{l-1}^m) + 2^m(t - t_{l-1}^m)\Delta_l^m h. \quad (40)$$

We shall quote several times algebraic identities set up in [4] for the processes  $W(m)$ ,  $m \geq 1$ , and replace  $W(m)$  by  $h(m)$ . Indeed, their proof rely only on the structure of the linear interpolations and not on the probabilistic properties of the fractional Brownian motion.

Our next purpose is to prove the convergence stated in (11). By the inequality (12), this amounts to prove the next proposition.

**Proposition 4** *Let  $p \in ]1, 4[$  be such that  $pH > 1$  and  $\alpha > 0$ . Then,*

(a) *For every  $j = 1, 2, 3$ ,*

$$\lim_{m, m' \rightarrow \infty} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} D_{j,p}(h(m), h(m')) = 0. \quad (41)$$

(b) *For every  $j = 1, 2$ ,*

$$\sup_{m \in \mathbb{N}} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} D_{j,p}(h(m)) < \infty. \quad (42)$$

**Proof:** *First order terms.* Let  $j = 1$  and  $k$  the index satisfying (15). By Lemma 3,

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} (D_{1,p}(h(m), h)^p) &= \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \sum_{n=m+1}^{\infty} n^\gamma \sum_{l=1}^{2^n} |2^{m-n} \Delta_k^m h - \Delta_l^n h|^p \\ &\leq C\alpha^p \sum_{n=m+1}^{\infty} n^\gamma \sum_{l=1}^{2^n} (2^{-mp(H-1)-np} + 2^{-npH}) \\ &\leq C\alpha^p 2^{-m(pH-1-\epsilon)}, \end{aligned}$$

with  $\epsilon \in ]0, pH - 1[$ . Hence, (41) holds for  $j = 1$ .

Similarly, for any  $\epsilon \in ]0, pH - 1[$ ,

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} (D_{1,p}(h)^p) &= \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \sum_{n=1}^{\infty} n^\gamma \sum_{l=1}^{2^n} |\Delta_l^n h|^p \\ &\leq \alpha^p \sum_{n=1}^{\infty} 2^{-n(pH-1-\epsilon)} \leq C\alpha^p, \end{aligned}$$

which together with (41) for  $j = 1$  give (42) for  $j = 1$ .

*Second order terms.* Consider now the case  $j = 2$ . Assume first  $m \leq n$ . Following [4], equation (17), pg. 118 for  $w(m) := h(m)$ , and using Lemma 3, we have for  $m < n$ ,

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} &\left| h(m+1)_{t_{l-1}^n, t_l^n}^2 - h(m)_{t_{l-1}^n, t_l^n}^2 \right| \\ &\leq \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left( \left| h(m+1)_{t_{l-1}^n, t_l^n}^2 \right| + \left| h(m)_{t_{l-1}^n, t_l^n}^2 \right| \right) \\ &\leq C 2^{2(m-n)} \left( \left| (\Delta_k^{m+1} h)^{\otimes 2} \right| + \left| (\Delta_k^m h)^{\otimes 2} \right| \right) \\ &\leq C\alpha^2 2^{-2nH}. \end{aligned} \tag{43}$$

Notice that we have also proved that for every  $m \leq n$ ,

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(m)_{t_{l-1}^n, t_l^n}^2 \right| \leq C\alpha^2 2^{-2nH}. \tag{44}$$

From equation (19) in [4] pg. 118 and Lemma 3, we easily obtain

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(n+1)_{t_{l-1}^n, t_l^n}^2 \right| \leq C\alpha^2 2^{-2nH}.$$

Thus, the above upper bound (43) holds for any  $m \leq n$ .

Suppose now  $n < m$ . Quoting [4], equation (20), pg 118, we write

$$h(m+1)_{t_{l-1}^n, t_l^n}^2 - h(m)_{t_{l-1}^n, t_l^n}^2 = \frac{1}{2} \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} \left( \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h \right. \\ \left. - \Delta_{2k}^{m+1} h \otimes \Delta_{2k-1}^{m+1} h \right),$$

for any  $l = 1, \dots, 2^n$ .

Fix  $d \geq 2$  and components  $(i, j)$  of the tensor products with  $i \neq j$ . Clearly,

$$\left| h(m+1)_{t_{l-1}^n, t_l^n}^{2,i,j} - h(m)_{t_{l-1}^n, t_l^n}^{2,i,j} \right| \leq C \left( T_{m,n,l}^{i,j} + T_{m,n,l}^{j,i} \right),$$

with

$$T_{m,n,l}^{i,j} = \left| \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} \Delta_{2k-1}^{m+1} h^i \Delta_{2k}^{m+1} h^j \right| \\ = \left| \int_0^1 \int_0^1 \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} \left( K(t_{2k-1}^{m+1}, t) - K(t_{2k-2}^{m+1}, t) \right) \right. \\ \left. \times \left( K(t_{2k}^{m+1}, s) - K(t_{2k-1}^{m+1}, s) \right) \dot{h}^i(s) \dot{h}^j(t) ds dt \right|. \quad (45)$$

Schwarz's inequality yields

$$T_{m,n,l}^{i,j} \leq C \alpha^2 \left( T_{m,n,l}^{i,j}(1) + T_{m,n,l}^{i,j}(2) \right)^{\frac{1}{2}},$$

where

$$T_{m,n,l}^{i,j}(1) = \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} \int_0^1 \int_0^1 \left( K(t_{2k-1}^{m+1}, t) - K(t_{2k-2}^{m+1}, t) \right)^2 \\ \times \left( K(t_{2k}^{m+1}, s) - K(t_{2k-1}^{m+1}, s) \right)^2 ds dt, \\ T_{m,n,l}^{i,j}(2) = \sum_{\substack{k, k'=2^{m-n}(l-1)+1 \\ k < k'}}^{2^{m-n}l} \int_0^1 \int_0^1 \left( K(t_{2k-1}^{m+1}, t) - K(t_{2k-2}^{m+1}, t) \right) \\ \times \left( K(t_{2k'-1}^{m+1}, t) - K(t_{2k'-2}^{m+1}, t) \right) \left( K(t_{2k}^{m+1}, s) - K(t_{2k-1}^{m+1}, s) \right) \\ \times \left( K(t_{2k'}^{m+1}, s) - K(t_{2k'-1}^{m+1}, s) \right) ds dt.$$

Clearly,

$$T_{m,n,l}^{i,j}(1) = \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} E \left| \Delta_{2k-1}^{m+1} W_i \right|^2 E \left| \Delta_{2k}^{m+1} W_j \right|^2 \\ \leq C 2^{-m(4H-1)-n}. \quad (46)$$



Moreover,

$$T_{m,n,l}^{i,j}(2) = \sum_{\substack{k,k'=2^{m-n(l-1)+1} \\ k < k'}}^{2^{m-nl}} E\left(\Delta_{2k-1}^{m+1} W_i \Delta_{2k'-1}^{m+1} W_i\right) E\left(\Delta_{2k}^{m+1} W_j \Delta_{2k'}^{m+1} W_j\right).$$

Notice that, whenever  $k < k'$ ,  $2(k' - k) \geq 2$ . Hence, applying the upper bound set up in [4], equation (29), pg. 121, we obtain

$$\begin{aligned} T_{m,n,l}^{i,j}(2) &\leq C 2^{-4(m+1)H} \sum_{\substack{k,k'=2^{m-n(l-1)+1} \\ k < k'}}^{2^{m-nl}} |k' - k|^{4(H-1)} \\ &\leq C 2^{-4(m+1)H} 2^{m-n} \sum_{r=1}^{2^{m-n}} r^{-4(1-H)} \\ &\leq C 2^{-m(4H-1)-n}. \end{aligned} \quad (47)$$

From (46), (47), it follows that, if  $n < m$ ,

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(m+1)_{t_{l-1}^n, t_l^n}^2 - h(m)_{t_{l-1}^n, t_l^n}^2 \right| \leq C \alpha^2 2^{-m(2H-\frac{1}{2})-\frac{n}{2}}. \quad (48)$$

Putting together (43) (valid for  $m \leq n$ ) and (48), we obtain

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left( D_{2,p}(h(m+1), h(m)) \right)^{\frac{p}{2}} &\leq C \alpha^p \left( \sum_{n=1}^{m-1} n^\gamma 2^{n(1-\frac{p}{4})} 2^{-mp(H-\frac{1}{4})} \right. \\ &\quad \left. + \sum_{n=m}^{\infty} n^\gamma 2^{-n(pH-1)} \right). \end{aligned}$$

Let  $\epsilon \in ]0, pH - 1[$ . The above estimates show the existence of some positive real number  $\beta$  such that

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} D_{2,p}(h(m+1), h(m)) \leq C \alpha^2 2^{-m\beta}. \quad (49)$$

This yields (41) for  $j = 2$ . The proof of (42) for  $j = 2$  is an easy consequence of (49) and (44).

*Third order terms.* We finally prove (41) for  $j = 3$ ; note that these terms only appear when  $H \in ]\frac{1}{4}, \frac{1}{3}]$ , so that  $p \in ]3, 4[$ .

Assume first  $m \leq n$ . In this case, for any  $l = 1, \dots, 2^n$  and  $k$  satisfying (15)

$$h(m)_{t_{l-1}^n, t_l^n}^3 = \frac{2^{3(m-n)}}{3!} (\Delta_k^m h)^{\otimes 3}. \quad (50)$$

We shall check that

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(m+1)_{t_{l-1}^n, t_l^n}^3 - h(m)_{t_{l-1}^n, t_l^n}^3 \right| \leq C\alpha^3 2^{-3nH}. \quad (51)$$

Indeed, if  $m < n$ , owing to (50) and Lemma 3,

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(m+1)_{t_{l-1}^n, t_l^n}^3 - h(m)_{t_{l-1}^n, t_l^n}^3 \right| &\leq \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left( \left| h(m+1)_{t_{l-1}^n, t_l^n}^3 \right| \right. \\ &\quad \left. + \left| h(m)_{t_{l-1}^n, t_l^n}^3 \right| \right) \\ &\leq C\alpha^3 2^{-3m(H-1)-3n} \leq C\alpha^3 2^{-3nH}. \end{aligned}$$

For  $m = n$ , we write the analogue of (30) with  $W$  replaced by  $h$ . With Lemma 3, we can check that each term of the resulting formula is bounded above by  $C\alpha^3 2^{-(n+1)3H}$ . Consequently, (51) holds for any  $m \leq n$ .

Let us now assume  $n < m$ . We write the identity given in Lemma 11 in [4] with  $w$  replaced by  $h$ . More precisely,

$$\left| h(m+1)_{t_{l-1}^n, t_l^n}^3 - h(m)_{t_{l-1}^n, t_l^n}^3 \right| \leq C \sum_{r=1}^5 |I_r(m, n, l)|,$$

with

$$\begin{aligned} I_1(m, n, l) &= \sum_k \left( h(t_{2k-2}^{m+1}) - h(t_{l-1}^n) \right) \otimes \left( \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h - \Delta_{2k}^{m+1} h \otimes \Delta_{2k-1}^{m+1} h \right), \\ I_2(m, n, l) &= \sum_k \left( \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h - \Delta_{2k}^{m+1} h \otimes \Delta_{2k-1}^{m+1} h \right) \otimes \left( h(t_l^n) - h(t_{2k+2}^{m+1}) \right), \\ I_3(m, n, l) &= \sum_k \Delta_{2k-1}^{m+1} h \otimes \left( \Delta_{2k}^{m+1} h \otimes \Delta_{2k}^{m+1} h + \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h \right), \\ I_4(m, n, l) &= \sum_k \Delta_{2k}^{m+1} h \otimes \left( \Delta_{2k}^{m+1} h \otimes \Delta_{2k-1}^{m+1} h + \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h \right), \\ I_5(m, n, l) &= \sum_k \left( \Delta_{2k-1}^{m+1} h \otimes \Delta_{2k}^{m+1} h + \Delta_{2k}^{m+1} h \otimes \Delta_{2k-1}^{m+1} h \right) \otimes \Delta_{2k-1}^{m+1} h, \end{aligned}$$

where the index  $k$  in the sums runs in the set  $\{2^{m-n}(l-1) + 1, \dots, 2^{m-n}l\}$ . The first two terms above have the same structure; the last three ones are also similar. They shall be analysed separately.

We start with  $I_1(m, n, l)$ . Notice that if  $d = 1$ , this term vanishes. Moreover, for  $d \geq 2$  only the components  $I_1(m, n, l)^{\kappa, i, j}$  with  $i \neq j$  might not vanish.

Let  $i \neq j$ . Clearly,

$$\begin{aligned} \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| I_1(m, n, l)^{\kappa, i, j} \right| &\leq \sup_{\|h\|_{\mathcal{H}} \leq \alpha} \sup_k \left| h(t_{2k-2}^{m+1}) - h(t_{l-1}^n) \right| \\ &\quad \times \left( T_{m, n, l}^{i, j} + T_{m, n, l}^{j, i} \right), \end{aligned}$$

with  $T_{m,n,l}^{i,j}$  defined in (45). Then, Lemma 3 together with (46) and (47) yield

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| I_1(m, n, l)^{\kappa, i, j} \right| \leq C\alpha^3 2^{-m(2H-\frac{1}{2})-n(H+\frac{1}{2})}, \quad (52)$$

and the same estimate holds for  $\sup_{\|h\|_{\mathcal{H}} \leq \alpha} |I_2(m, n, l)^{\kappa, i, j}|$ .

Set

$$J(m, n, l) = \sum_{k=2^{m-n}(l-1)+1}^{2^{m-n}l} \Delta_{a(k)}^{m+1} h \otimes \Delta_{b(k)}^{m+1} h \otimes \Delta_{c(k)}^{m+1} h,$$

where  $a(k), b(k)$  and  $c(k)$  belong to  $\in \{2k-1, 2k\}$  and are such that two out of the three indices agree. Lemma 3 yields

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} |J(m, n, l)| \leq C\alpha^3 2^{-m(3H-1)-n},$$

which implies

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} |I_{\mu}(m, n, l)| \leq C\alpha^3 2^{-m(3H-1)-n}, \quad (53)$$

for any  $\mu = 3, 4, 5$ .

Since for  $n < m$ ,  $2^{-m(2H+\frac{1}{2})-n(H-\frac{1}{2})} < 2^{-m(3H-1)-n}$ , the estimates (52) and (53) imply

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} \left| h(m+1)_{t_{l-1}^n}^3 - h(m)_{t_{l-1}^n}^3 \right| \leq C\alpha^3 2^{-m(3H-1)-n}. \quad (54)$$

By the very definition of  $D_{3,p}(h(m+1), h(m))$  and taking into account the results obtained for  $m \leq n$  in (51) and for  $m > n$  in (54), we obtain

$$D_{3,p}(h(m+1), h(m))^{\frac{p}{3}} \leq C\alpha^p \left( \sum_{n=1}^{m-1} n^{\gamma} 2^{-mp(H-\frac{1}{3})-n\frac{p}{3}} + \sum_{n=m}^{\infty} n^{\gamma} 2^{-n(pH-1)} \right).$$

Since  $p > 3$ , this yields

$$\sup_{\|h\|_{\mathcal{H}} \leq \alpha} D_{3,p}(h(m+1), h(m)) \leq C\alpha^3 2^{-m\beta},$$

for some real  $\beta > 0$ . This suffices to establish (41) for  $j = 3$  and ends the proof of the proposition.  $\square$

**Remark:** For  $H \in ]\frac{1}{2}, 1[$ ,  $F(W) = (1, W^1)$  is a geometric rough path of roughness  $p$ , with  $pH > 1$ . The large deviation principle stated in Theorem 1 also holds for these values of the parameters  $H$  and  $p$ . Indeed, it is a consequence of (13) and (41) for  $j = 1$ .

### 3 Geometric rough paths on the reproducing kernel Hilbert space

Proposition 4 implies the existence of a geometric rough path of roughness  $p$ , for any  $p \in ]1, 4[$  with  $pH > 1$ , lying above  $h \in \mathcal{H}$ . In this section we give a representation of this object in terms of multiple integrals based on  $h$ .

We start by introducing the type of integrals to be used. They are a sort of deterministic counterpart of the stochastic integral with respect to the fractional Brownian motion introduced in [1] (see also [2]).

Following [1], for a function  $\varphi : [0, 1] \rightarrow \mathbb{R}$  we set

$$\|\varphi\|_K^2 = \int_0^1 \varphi(s)^2 K(1, s)^2 ds + \int_0^1 ds \left( \int_s^1 |\varphi(t) - \varphi(s)| |K|(dt, s) \right)^2, \quad (55)$$

$$K^*(\varphi \mathbf{1}_{[0, t]})(s) = \varphi(s)K(t, s) + \int_s^t (\varphi(r) - \varphi(s))K(dr, s). \quad (56)$$

Notice that,  $\|\varphi\|_K < \infty$  implies  $\|\varphi \mathbf{1}_{[0, t]}\|_K < \infty$  as well, for any  $t \in [0, 1]$ . We denote by  $\mathcal{H}_K$  the completion of the set  $\mathcal{E}$  of step functions on  $[0, 1]$  with respect to the semi-norm  $\|\cdot\|_K$ .

In the sequel, we set  $(K\dot{h})(t) = h(t)$  for any  $h \in \mathcal{H}$  with representation given in (3) in terms of  $\dot{h} \in L^2([0, 1])$ . By Lemma 1 in [1], for any step function  $\varphi \in \mathcal{E}$ , we have

$$\int_0^1 \varphi(t)h(dt) = \int_0^1 \varphi(t)(K\dot{h})(dt) = \int_0^1 K^*(\varphi)(t)\dot{h}(t)dt. \quad (57)$$

Thus, the linear continuous functional  $\varphi \mapsto \int_0^1 \varphi(t)h(dt)$  defined on  $\mathcal{E}$  - endowed with the topology induced by the semi-norm  $\|\cdot\|_K$  - taking values in  $\mathbb{R}$ , can be extended to  $\mathcal{H}_K$ . Hence, we attach a meaning to the indefinite integral of  $\varphi \in \mathcal{H}_K$  with respect to  $h \in \mathcal{H}$  by means of the formula

$$\int_0^t \varphi(s)h(ds) = \int_0^1 K^*(\varphi \mathbf{1}_{[0, t]})(s) \dot{h}(s)ds. \quad (58)$$

The following lemma establishes the existence of the indefinite multiple Itô-Wiener integral with respect to the fractional Brownian motion and its continuity. Recall that we set  $K(t, s) = 0$  for  $s \geq t$ .

**Proposition 5** *Let  $\varphi$  be a Hölder continuous real-valued function defined on  $[0, 1]$ , of order  $\lambda \in (0, 1)$  with  $\lambda + H > \frac{1}{2}$ . Then  $\varphi \in \mathcal{H}_K$  and the function  $t \mapsto \int_0^t \varphi(s)h(ds)$  is Hölder continuous of order  $H$ .*

**Proof:** First we prove that  $\|\varphi\|_K < \infty$ . Clearly,

$$\int_0^1 \varphi(s)^2 K(1, s)^2 ds \leq \|\varphi\|_\infty^2 \int_0^1 K(1, s)^2 ds = \|\varphi\|_\infty^2 < \infty,$$

with  $\|\varphi\|_\infty = \sup_{t \in [0,1]} |\varphi(t)|$ . Moreover, (5) implies

$$\int_0^1 ds \left( \int_s^1 |\varphi(t) - \varphi(s)| |K|(dt, s) \right)^2 \leq C \int_0^1 ds \left( \int_s^1 |t - s|^{\lambda+H-\frac{3}{2}} dt \right)^2 < \infty.$$

The two above inequalities yield  $\|\varphi\|_K < \infty$ .

For any  $m \geq 1$ , we consider the step function  $\varphi_m(s) = \sum_{l=1}^{2^m} \mathbf{1}_{\Delta_l^m}(s) \varphi(t_{l-1}^m)$ . Since  $\varphi$  is Hölder continuous,

$$\sup_{l=1, \dots, 2^m} \sup_{s \in \Delta_l^m} |\varphi_m(s) - \varphi(s)| \leq C 2^{-\lambda m}.$$

Consequently,

$$\lim_{m \rightarrow \infty} \int_0^1 |\varphi_m(s) - \varphi(s)|^2 K(1, s)^2 ds \leq \lim_{m \rightarrow \infty} C 2^{-2\lambda m} \int_0^1 K(1, s)^2 ds = 0.$$

Moreover,

$$\lim_{m \rightarrow \infty} \left| (\varphi_m(t) - \varphi_m(s)) - (\varphi(t) - \varphi(s)) \right| \leq C \lim_{m \rightarrow \infty} 2^{-\lambda m} = 0,$$

and

$$\sup_{m \geq 1} \left| (\varphi_m(t) - \varphi_m(s)) - (\varphi(t) - \varphi(s)) \right| \leq C |t - s|^\lambda,$$

whenever  $s \in \Delta_l^m, t \in \Delta_{l'}^m$ , with  $|l - l'| > 1$ .

Set

$$I_m(\varphi) = \int_0^1 ds \left( \int_s^1 \left| (\varphi_m(t) - \varphi_m(s)) - (\varphi(t) - \varphi(s)) \right| |K|(dt, s) \right)^2$$

and  $I_m(\varphi) \leq C \sum_{i=1}^3 I_m^i(\varphi)$  with

$$I_m^1(\varphi) = \sum_{l=1}^{2^m} \int_{\Delta_l^m} ds \left( \int_{t_{l+1}^m}^1 \left| (\varphi_m(t) - \varphi_m(s)) - (\varphi(t) - \varphi(s)) \right| |K|(dt, s) \right)^2,$$

$$I_m^2(\varphi) = \sum_{l=1}^{2^m} \int_{\Delta_l^m} ds \left( \int_s^{t_l^m} |\varphi(t) - \varphi(s)| |K|(dt, s) \right)^2,$$

$$I_m^3(\varphi) = \sum_{l=1}^{2^m} \int_{\Delta_l^m} ds \left( \int_{\Delta_{l+1}^m} \left| (\varphi_m(t) - \varphi_m(s)) - (\varphi(t) - \varphi(s)) \right| |K|(dt, s) \right)^2.$$

By the bounded convergence theorem applied first to the integral with respect to the measure  $|K|(dt, s)$  and then to the Lebesgue measure, we have  $\lim_{m \rightarrow \infty} I_m^1(\varphi) = 0$ .

Moreover,

$$\begin{aligned} I_m^2(\varphi) &\leq C \sum_{l=1}^{2^m} \int_{\Delta_l^m} ds \left( \int_s^{t_l^m} |t-s|^{\lambda+H-\frac{3}{2}} dt \right)^2 \\ &\leq C 2^{-m(2\lambda+2H-1)}. \end{aligned}$$

Thus,  $\lim_{m \rightarrow \infty} I_m^2(\varphi) = 0$ .

Since

$$\sup_{s, t \in [0, 1]} (|\varphi_m(t) - \varphi(t)| + |\varphi_m(s) - \varphi(s)|) \leq C 2^{-\lambda m},$$

it follows that

$$\begin{aligned} I_m^3(\varphi) &\leq C \sum_{l=1}^{2^m} 2^{-2\lambda m} \int_{\Delta_l^m} ds \left( \int_{\Delta_{l+1}^m} |t-s|^{H-\frac{3}{2}} dt \right)^2 \\ &\leq C 2^{-m(2\lambda+2H-1)}, \end{aligned}$$

and therefore,  $\lim_{m \rightarrow \infty} I_m^3(\varphi) = 0$ .

Therefore,  $\lim_{m \rightarrow \infty} I_m(\varphi) = 0$  and we have thus established that  $\varphi \in \mathcal{H}_K$ .

Let us now prove the Hölder continuity of the indefinite integral  $\int_0^t \varphi(s)h(ds)$ . Fix  $0 \leq t_1 \leq t_2 \leq 1$ . By virtue of (58) and (56),

$$\int_0^{t_2} \varphi(s)h(ds) - \int_0^{t_1} \varphi(s)h(ds) = \sum_{i=1}^3 T_i(t_1, t_2),$$

with

$$\begin{aligned} T_1(t_1, t_2) &= \int_0^{t_1} ds \dot{h}(s) \left( \int_{t_1}^{t_2} \varphi(r)K(dr, s) \right), \\ T_2(t_1, t_2) &= \int_{t_1}^{t_2} \varphi(s)K(t_2, s) \dot{h}(s) ds, \\ T_3(t_1, t_2) &= \int_{t_1}^{t_2} ds \dot{h}(s) \left( \int_s^{t_2} (\varphi(r) - \varphi(s))K(dr, s) \right). \end{aligned}$$

Schwarz's inequality yields

$$\begin{aligned} |T_1(t_1, t_2)| &\leq \|\varphi\|_\infty \|\dot{h}\|_2 \left( \int_0^{t_1} ds \left( \int_{t_1}^{t_2} |K|(dr, s) \right)^2 \right)^{\frac{1}{2}} \\ &\leq \|\varphi\|_\infty \|\dot{h}\|_2 \left( \int_0^{t_1} |K(t_2, s) - K(t_1, s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \|\varphi\|_\infty \|\dot{h}\|_2 |t_2 - t_1|^H. \end{aligned} \tag{59}$$

Similarly,

$$\begin{aligned} |T_2(t_1, t_2)| &\leq \|\varphi\|_\infty \|\dot{h}\|_2 \left( \int_{t_1}^{t_2} K(t_2, s)^2 ds \right)^{\frac{1}{2}} \\ &\leq \|\varphi\|_\infty \|\dot{h}\|_2 |t_2 - t_1|^H. \end{aligned} \quad (60)$$

The Hölder continuity of the function  $\varphi$  together with the upper bound given in (5), imply

$$\begin{aligned} |T_3(t_1, t_2)| &\leq C \|\dot{h}\|_2 \left( \int_{t_1}^{t_2} ds \left( \int_s^{t_2} |r - s|^{\lambda+H-\frac{3}{2}} dr \right)^2 \right)^{\frac{1}{2}} \\ &\leq C \|\dot{h}\|_2 |t_2 - t_1|^{\lambda+H}. \end{aligned} \quad (61)$$

With (59)–(61), we have

$$\left| \int_0^{t_2} \varphi(s) h(ds) - \int_0^{t_1} \varphi(s) h(ds) \right| \leq C \|\dot{h}\|_2 |t_2 - t_1|^H.$$

This completes the proof of the proposition.  $\square$

The preceding proposition provides a background to define indefinite iterated integrals with respect to elements of the reproducing kernel Hilbert space of the fractional Brownian motion, as follows.

**Corollary 6** *The reproducing kernel Hilbert space  $\mathcal{H}$  of the fractional Brownian motion with Hurst parameter  $H \in ]\frac{1}{4}, \frac{1}{2}[$  is contained in  $\mathcal{H}_K$ . Given  $h \in \mathcal{H}$ , the indefinite integral  $h_{0,t}^2 := \int_0^t h(s)h(ds)$  defines a  $H$ -Hölder continuous function. Therefore, the function  $t \mapsto h_{0,t}^2$  belongs to  $\mathcal{H}_K$ . Thus, it can be integrated again with respect to  $h$ . The resulting integral inherits the  $H$ -Hölder continuity property.*

Let  $g$  be a measurable Lebesgue integrable function defined on  $[0, 1]$ . For  $l \in \{1, \dots, 2^m\}$ , set

$$a_l^m(t) = 2^m \int_{\Delta_l^m \cap [0,t]} g(s) ds.$$

Consider the linear interpolation of  $h$ , that is the function  $h(m)$  defined in (40). Obviously,

$$\begin{aligned} \int_0^t g(s) h(m)(ds) &= \sum_{l=1}^{[2^m t]+1} a_l^m(t) \Delta_l^m h \\ &= \sum_{l=1}^{[2^m t]+1} a_l^m(t) \left( (K\dot{h})(t_l^m) - (K\dot{h})(t_{l-1}^m) \right). \end{aligned} \quad (62)$$

Following the steps of the proof of Lemma 1 in [1], consisting actually into an integration by parts, we obtain

$$\int_0^t g(s)h(m)(ds) = \int_0^1 K(m)^*(g\mathbf{1}_{[0,t]})(s) \dot{h}(s)ds, \quad (63)$$

with

$$\begin{aligned} K(m)^*(g\mathbf{1}_{[0,t]})(s) &= \sum_{l=1}^{[2^m t]+1} \mathbf{1}_{\Delta_l^m}(s) a_l^m(t) K(t_{[2^m t]+1}^m, s) \\ &+ \sum_{l=1}^{[2^m t]} \mathbf{1}_{\Delta_l^m}(s) \sum_{l'=l+1}^{[2^m t]+1} (a_{l'}^m(t) - a_l^m(t)) (K(t_{l'}^m, s) - K(t_{l'-1}^m, s)). \end{aligned} \quad (64)$$

Notice the similarity between the expressions (64) and (56).

Our next aim is to prove that  $\lim_{m \rightarrow \infty} \int_0^t G(m)(s)h(m)(ds) = \int_0^t G(s)h(ds)$ , for the pairs  $G(m) = h(m)$ ,  $G = h$ , and  $G(m) = \int_0^{\cdot} h(m)(s)h(m)(ds)$ ,  $G = \int_0^{\cdot} h(s)h(ds)$ , respectively. As a consequence we shall obtain in Theorem 9 an integral expression for the geometric rough path lying above  $h$ . A basic ingredient of its proof is provided by the next statement.

**Proposition 7** *Let  $g$  be a  $\lambda$ -Hölder continuous real-valued function defined on  $[0, 1]$ , with  $\lambda + H > \frac{1}{2}$ . Then, there exists a constant  $C > 0$  such that for any  $0 \leq t_1 \leq t_2 \leq 1$ ,*

$$\sup_{m \in \mathbb{N}} \left| \int_0^{t_2} g(s)h(m)(ds) - \int_0^{t_1} g(s)h(m)(ds) \right| \leq C |t_2 - t_1|^H. \quad (65)$$

*In particular, each indefinite integral  $\int_0^{\cdot} g(s)h(m)(ds)$  defines a  $H$ -Hölder continuous function.*

**Proof:** Fix  $m \geq 1$ . Assume first that  $[2^m t_1] = [2^m t_2]$ , so that  $|t_2 - t_1| \leq 2^{-m}$ . Owing to (62),

$$\begin{aligned} \left| \int_0^{t_2} g(s)h(m)(ds) - \int_0^{t_1} g(s)h(m)(ds) \right| &= \left| 2^m \left( \int_{t_1}^{t_2} g(r) dr \right) \right. \\ &\quad \left. \times \int_0^1 \left( K(t_{[2^m t_1]+1}^m, s) - K(t_{[2^m t_1]}^m, s) \right) \dot{h}(s) ds \right| \\ &\leq \|g\|_\infty \|\dot{h}\|_2 2^{-m(H-1)} |t_2 - t_1| \leq C |t_2 - t_1|^H. \end{aligned} \quad (66)$$

Suppose now that  $[2^m t_1] < [2^m t_2]$ . Then using (62) we have

$$\left| \int_0^{t_2} g(s)h(m)(ds) - \int_0^{t_1} g(s)h(m)(ds) \right| \leq \sum_{j=1}^3 \left| \int_0^1 S_j(t_1, t_2, s) \dot{h}(s) ds \right|,$$



with

$$\begin{aligned}
S_1(t_1, t_2, s) &= \left( 2^m \int_{t_1}^{t_{[2^m t_1]+1}^m} g(r) dr \right) \left( K(t_{[2^m t_1]+1}^m, s) - K(t_{[2^m t_1]}^m, s) \right), \\
S_2(t_1, t_2, s) &= \left( 2^m \int_{t_{[2^m t_2]}^m}^{t_2} g(r) dr \right) \left( K(t_{[2^m t_2]+1}^m, s) - K(t_{[2^m t_2]}^m, s) \right), \\
S_3(t_1, t_2, s) &= \sum_{l=[2^m t_1]+2}^{[2^m t_2]} a_l^m(t_2) \Delta_l^m K(\cdot, s),
\end{aligned}$$

with the convention that  $\sum_{l=I}^J x_l = 0$  if  $I > J$ .

The arguments used to prove (66) show that

$$\sum_{j=1}^2 \left| \int_0^1 S_j(t_1, t_2, s) \dot{h}(s) ds \right| \leq C |t_2 - t_1|^H. \quad (67)$$

The inequalities (66) and (67) prove (65) if  $[2^m t_2] = [2^m t_1] + 1$ . In order to conclude the proof, assume that  $[2^m t_2] \geq [2^m t_1] + 2$  and let us estimate  $S_3(t_1, t_2, s)$ .

Following again the steps of the proof of Lemma 1 in [1], we deduce  $S_3(t_1, t_2, s) = \sum_{j=1}^3 S_{3,j}(t_1, t_2, s)$ , with

$$\begin{aligned}
S_{3,1}(t_1, t_2, s) &= 1_{]0, t_{[2^m t_1]+1}^m[}(s) \sum_{l=[2^m t_1]+2}^{[2^m t_2]} a_l^m(t_2) \Delta_l^m K(\cdot, s), \\
S_{3,2}(t_1, t_2, s) &= \sum_{l=[2^m t_1]+2}^{[2^m t_2]} 1_{\Delta_l^m}(s) a_l^m(t_2) K(t_{[2^m t_2]}^m, s), \\
S_{3,3}(t_1, t_2, s) &= \sum_{l=[2^m t_1]+2}^{[2^m t_2]-1} 1_{\Delta_l^m}(s) \left( \sum_{l'=l+1}^{[2^m t_2]} (a_{l'}^m(t_2) - a_l^m(t_2)) \Delta_{l'}^m K(\cdot, s) \right).
\end{aligned}$$

Note that this decomposition is similar to that used to prove the Hölder regularity of the indefinite stochastic integral  $\int_0^\cdot \varphi(s) h(ds)$ . Actually, out of the factor  $\dot{h}$ ,  $S_{3,j}(t_1, t_2, s)$ ,  $j = 1, 2, 3$ , are the analogue of the integrands of  $T_j(t_1, t_2)$ ,  $j = 1, 2, 3$ , respectively.

By (5), we have that the function  $t \mapsto K(t, s)$  is decreasing on  $]s, 1]$ . Hence, given  $1 \leq I < J$  and  $s \leq t_{I-1}^m$ ,

$$\sum_{l=I}^J \left| \Delta_l^m K(\cdot, s) \right| = \left| K(t_J^m, s) - K(t_{I-1}^m, s) \right|.$$

Since  $\sup_{m,l,t} |a_l^m(t)| \leq \|g\|_\infty$ , we have

$$\begin{aligned} \left| \int_0^1 S_{3,1}(t_1, t_2, s) \dot{h}(s) ds \right| &\leq \|g\|_\infty \|\dot{h}\|_2 \\ &\times \left( \int_0^{t_{[2^m t_1]+1}^m} ds \left| K(t_{[2^m t_2]}^m, s) - K(t_{[2^m t_1]+1}^m, s) \right|^2 \right)^{\frac{1}{2}} \\ &\leq C \left| t_{[2^m t_2]}^m - t_{[2^m t_1]+1}^m \right|^H \leq C |t_2 - t_1|^H \end{aligned} \quad (68)$$

and

$$\begin{aligned} \left| \int_0^1 S_{3,2}(t_1, t_2, s) \dot{h}(s) ds \right| &\leq \|g\|_\infty \|\dot{h}\|_2 \left( \int_{t_{[2^m t_1]+1}^m}^{t_{[2^m t_2]}^m} ds \left| K(t_{[2^m t_2]}^m, s) \right|^2 \right)^{\frac{1}{2}} \\ &\leq C \left| t_{[2^m t_2]}^m - t_{[2^m t_1]+1}^m \right|^H \leq C |t_2 - t_1|^H. \end{aligned} \quad (69)$$

The Hölder continuity of  $g$  implies that for  $s \in \Delta_l^m$ ,  $r \in \Delta_{l'}^m$  with  $[2^m t_1] + 2 \leq l < l' \leq [2^m t_2]$ ,  $|a_{l'}^m(t_2) - a_l^m(t_2)| \leq C ((l' - l)2^{-m})^\lambda \leq C(2^{-m\lambda} 1_{\{l'=l+1\}} + |r - s|^\lambda 1_{\{l' > l+1\}})$ . Therefore, since  $|t_2 - t_1| \geq 2^{-m}$ ,

$$\begin{aligned} \left| \int_0^1 S_{3,3}(t_1, t_2, s) \dot{h}(s) ds \right| &\leq C \|\dot{h}\|_2 \left( \int_0^1 ds \sum_{l=[2^m t_1]+2}^{[2^m t_2]-1} 1_{\Delta_l^m}(s) \right. \\ &\quad \times \left. \left( \left[ \int_{t_{l+1}^m}^{t_{[2^m t_2]}^m} |r - s|^{\lambda+H-\frac{3}{2}} dr \right]^2 + \left[ 2^{-m\lambda} \int_{t_l^m}^{t_{l+1}^m} |r - s|^{H-\frac{3}{2}} dr \right]^2 \right) \right)^{\frac{1}{2}} \\ &\leq C \|\dot{h}\|_2 \left( \int_0^1 ds \sum_{l=[2^m t_1]+2}^{[2^m t_2]-1} 1_{\Delta_l^m}(s) \left[ (t_{[2^m t_2]}^m - s)^{2\lambda+2H-1} \right. \right. \\ &\quad \left. \left. + 2^{-2m\lambda} (t_l^m - s)^{2H-1} \right] \right)^{\frac{1}{2}} \\ &\leq C \|\dot{h}\|_2 \left( |t_2 - t_1|^{\lambda+H} + 2^{-m(\lambda+H-\frac{1}{2})} |t_2 - t_1|^{\frac{1}{2}} \right) \\ &\leq C \|\dot{h}\|_2 |t_2 - t_1|^{\lambda+H}. \end{aligned} \quad (70)$$

The inequalities (67)-(70) conclude the proof of the proposition.  $\square$

We next prove the announced result on convergence of integrals.

**Proposition 8** *Let  $h = K\dot{h} \in \mathcal{H}$ ,  $G(m)$ ,  $m \geq 1$ , and  $G$  be real continuous functions defined on  $[0, 1]$ . Assume that for any  $0 \leq t_1 \leq t_2 \leq 1$ ,*

$$(i) \quad |G(t_2) - G(t_1)| \leq C |t_2 - t_1|^H,$$

$$(ii) \sup_{m \in \mathbb{N}} |G(m)(t_2) - G(m)(t_1)| \leq C |t_2 - t_1|^H,$$

$$(iii) c(m) := \sup_{t \in [0,1]} |G(m)(t) - G(t)| \rightarrow 0, \text{ as } m \rightarrow \infty.$$

Then

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \left| \int_0^t G(m)(s) h(m)(ds) - \int_0^t G(s) h(ds) \right| = 0.$$

**Proof:** Set  $A_l^m(t) = 2^m \int_{\Delta_l^m \cap [0,t]} G(m)(s) ds$ . We first prove that

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \left| \int_0^t ds \dot{h}(s) \left( \sum_{l=1}^{[2^m t]+1} \mathbf{1}_{\Delta_l^m}(s) A_l^m(t) K(t_{[2^m t]+1}^m, s) \right) - G(s) K(t, s) \right| = 0. \quad (71)$$

Indeed, Schwarz's inequality yields

$$\begin{aligned} & \left| \int_0^t ds \dot{h}(s) \sum_{l=1}^{[2^m t]+1} \mathbf{1}_{\Delta_l^m}(s) G(s) \left( K(t_{[2^m t]+1}^m, s) - K(t, s) \right) \right| \\ & \leq \|\dot{h}\|_2 \|G\|_\infty \left( \int_0^1 ds \sum_{l=1}^{[2^m t]+1} \mathbf{1}_{\Delta_l^m}(s) \left| K(t_{[2^m t]+1}^m, s) - K(t, s) \right|^2 \right)^{\frac{1}{2}} \\ & \leq C 2^{-mH}. \end{aligned} \quad (72)$$

Owing to (ii) and (iii),

$$\sup_{1 \leq l \leq [2^m t]} \sup_{r, s \in \Delta_l^m} |G(m)(r) - G(s)| \leq C(2^{-mH} + c(m)).$$

Therefore,

$$\begin{aligned} & \left| \int_0^t ds \dot{h}(s) K(t_{[2^m t]+1}^m, s) \left( \sum_{l=1}^{[2^m t]} \mathbf{1}_{\Delta_l^m}(s) \left( A_l^m(t) - 2^m \int_{\Delta_l^m} G(s) dr \right) \right) \right| \\ & \leq C(2^{-mH} + c(m)) \int_0^1 |\dot{h}(s)| |K(t_{[2^m t]+1}^m, s)| ds \\ & \leq C \|\dot{h}\|_2 (2^{-mH} + c(m)). \end{aligned} \quad (73)$$

By (ii) and (iii),

$$\sup_{r, s \in \Delta_{[2^m t]+1}^m} \left| G(s) - 2^m \int_{t_{[2^m t]}^m}^t G(m)(r) dr \right| \leq C(2^{-mH} + c(m) + \|G\|_\infty) \leq C.$$

Hence,

$$\begin{aligned} & \left| \int_0^t ds \dot{h}(s) K(t_{[2^m t]+1}^m, s) \left( \mathbf{1}_{\Delta_{[2^m t]+1}^m}(s) \left( 2^m \int_{t_{[2^m t]}^m}^t G(m)(r) dr - G(s) \right) \right) \right| \\ & \leq C \int_0^1 \mathbf{1}_{\Delta_{[2^m t]+1}^m}(s) |\dot{h}(s)| |K(t_{[2^m t]+1}^m, s)| ds \leq C 2^{-mH}. \end{aligned} \quad (74)$$

With (72)–(74), we have proved (71).

The second step of the proof consists in checking that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \left| \int_0^t ds \dot{h}(s) \left( \int_s^t (G(r) - G(s)) K(dr, s) \right. \right. \\ & \quad \left. \left. - \left[ \sum_{l=1}^{[2^m t]} 1_{\Delta_l^m}(s) \sum_{l'=l+1}^{[2^m t]+1} (A_{l'}^m(t) - A_l^m(t)) (K(t_{l'}^m, s) - K(t_{l'-1}^m, s)) \right] \right) \right| = 0. \end{aligned} \quad (75)$$

Clearly,

$$\int_0^t ds \dot{h}(s) \left( \int_s^t (G(r) - G(s)) K(dr, s) \right) = \sum_{i=1}^3 R_i^m(t),$$

with

$$\begin{aligned} R_1^m(t) &= \int_{\Delta_{[2^m t]+1}^m \cap [0,t]} ds \dot{h}(s) \left( \int_s^t (G(r) - G(s)) K(dr, s) \right), \\ R_2^m(t) &= \sum_{l=1}^{[2^m t]} \int_0^t ds 1_{\Delta_l^m}(s) \dot{h}(s) \left( \int_s^{t_l^m} (G(r) - G(s)) K(dr, s) \right), \\ R_3^m(t) &= \sum_{l=1}^{[2^m t]} \int_0^t ds 1_{\Delta_l^m}(s) \dot{h}(s) \left( \sum_{l'=l+1}^{[2^m t]+1} \int_{t_{l'-1}^m}^{t_{l'}^m} (G(r) - G(s)) 1_{\{s \leq r \leq t\}} K(dr, s) \right). \end{aligned}$$

By virtue of Schwarz's inequality, assumption (i) and (5) we have

$$\begin{aligned} |R_1^m(t)| &\leq C \|\dot{h}\|_2 \left( \int_{\Delta_{[2^m t]+1}^m \cap [0,t]} ds \left( \int_s^t |r - s|^{2H - \frac{3}{2}} dr \right)^2 \right)^{\frac{1}{2}} \\ &\leq C \|\dot{h}\|_2 2^{-2mH}. \end{aligned}$$

Using again (i) and (5) we obtain

$$\left| \int_s^{t_l^m} (G(r) - G(s)) K(dr, s) \right| \leq C (t_l^m - s)^{2H - \frac{1}{2}}.$$

It follows that

$$\sup_{l=1, \dots, [2^m t]} \sup_{s \in \Delta_l^m} \left| \int_s^{t_l^m} (G(r) - G(s)) K(dr, s) \right| \leq C 2^{-m(2H - \frac{1}{2})}$$

and consequently

$$|R_2^m(t)| \leq C 2^{-m(2H - \frac{1}{2})} \sum_{l=1}^{[2^m t]} \int_0^t ds 1_{\Delta_l^m}(s) |\dot{h}(s)| \leq C 2^{-m(2H - \frac{1}{2})}.$$

Thus we have shown

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} (|R_1^m(t)| + |R_2^m(t)|) = 0.$$

Therefore, the proof of (75) reduces to that of

$$\begin{aligned} \lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \left| \sum_{l=1}^{[2^m t]} \int_0^t ds \mathbf{1}_{\Delta_l^m}(s) \dot{h}(s) \left( \sum_{l'=l+1}^{[2^m t]+1} \left( \int_{t_{l'-1}^m}^{t_{l'}^m} (G(r) - G(s)) \mathbf{1}_{\{s \leq r \leq t\}} \right. \right. \right. \\ \left. \left. \left. - (A_{l'}^m(t) - A_l^m(t)) \right) K(dr, s) \right) \right| = 0. \end{aligned} \quad (76)$$

Set

$$\begin{aligned} R_4^m(t) &= \sum_{l=1}^{[2^m t]} \int_0^t ds \int_0^1 K(dr, s) \dot{h}(s) \mathbf{1}_{\Delta_l^m}(s) \mathbf{1}_{\Delta_{l+1}^m}(r) \\ &\quad \times \left[ (G(r) - G(s)) \mathbf{1}_{\{s \leq r \leq t\}} - (A_{l+1}^m(t) - A_l^m(t)) \right]. \end{aligned}$$

The Hölder continuity of  $G$  together with (5) implies

$$\begin{aligned} \left| \sum_{l=1}^{[2^m t]} \int_{\Delta_l^m} ds \dot{h}(s) \int_{\Delta_{l+1}^m} K(dr, s) (G(r) - G(s)) \mathbf{1}_{\{s \leq r \leq t\}} \right| \\ \leq C \left| \sum_{l=1}^{[2^m t]} \int_{\Delta_l^m} ds \dot{h}(s) \int_{\Delta_{l+1}^m} |r - s|^{2H - \frac{3}{2}} dr \right| \leq C \|\dot{h}\|_2 2^{-m(2H - \frac{1}{2})}. \end{aligned} \quad (77)$$

By assumption (ii), for any  $l = 1, \dots, [2^m t] - 1$ , we have

$$\sup_{t \in [0,1]} |A_{l+1}^m(t) - A_l^m(t)| \leq C 2^{-mH},$$

while for  $l = [2^m t]$ ,

$$\sup_{t \in [0,1]} |A_{l+1}^m(t) - A_l^m(t)| \leq C (2^{-mH} + c(m) + \|G\|_\infty) \leq C.$$

Therefore,

$$\begin{aligned}
& \left| \sum_{l=1}^{[2^m t]} \int_{\Delta_l^m} ds \dot{h}(s) \int_{\Delta_{l+1}^m} K(dr, s) (A_{l+1}^m(t) - A_l^m(t)) \right| \\
& \leq C 2^{-mH} \sum_{l=1}^{[2^m t]-1} \int_{\Delta_l^m} ds |\dot{h}(s)| \int_{\Delta_{l+1}^m} dr |r - s|^{H-\frac{3}{2}} \\
& \quad + C \int_{\Delta_{[2^m t]}^m} ds |\dot{h}(s)| \int_{\Delta_{[2^m t]+1}^m} dr |r - s|^{H-\frac{3}{2}} \\
& \leq C 2^{-mH} \|\dot{h}\|_2 \left( \sum_{l=1}^{[2^m t]-1} \int_{\Delta_l^m} ds |t_l^m - s|^{2H-1} \right)^{\frac{1}{2}} + C \|\dot{h}\|_2 2^{-mH} \\
& \leq C \|\dot{h}\|_2 \left( 2^{-\frac{m}{2}(4H-1)} + 2^{-mH} \right). \tag{78}
\end{aligned}$$

From (77) and (78) we obtain

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} |R_4^m(t)| = 0.$$

Consequently, it remains to prove that

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \left| \sum_{l=1}^{[2^m t]-1} \int_0^t ds \mathbf{1}_{\Delta_l^m}(s) \dot{h}(s) \left( \sum_{l'=l+2}^{[2^m t]+1} \left( \int_{\Delta_{l'}^m} ((G(r) - G(s)) \mathbf{1}_{\{s \leq r \leq t\}} \right. \right. \right. \right. \\
& \quad \left. \left. \left. - (A_{l'}^m(t) - A_l^m(t)) \right) K(dr, s) \right) \right| = 0. \tag{79}
\end{aligned}$$

Set

$$\Psi_m(t; s, r) = \sum_{l=1}^{[2^m t]-1} \sum_{l'=l+2}^{[2^m t]} \mathbf{1}_{\Delta_l^m}(s) \mathbf{1}_{\Delta_{l'}^m}(r) \left( (G(r) - G(s)) - (A_{l'}^m(t) - A_l^m(t)) \right).$$

A simple analysis based on the hypotheses (i) – (iii) gives

$$\begin{aligned}
& \sup_{l=1, \dots, [2^m t]-1} \left| \mathbf{1}_{\Delta_l^m}(s) (G(s) - A_l^m(t)) \right| + \sup_{l'=l+2, \dots, [2^m t]} \left| \mathbf{1}_{\Delta_{l'}^m}(r) (G(r) - A_{l'}^m(t)) \right| \\
& \leq C(2^{-mH} + c(m)).
\end{aligned}$$

Thus

$$|\Psi_m(t; s, r)| \leq C(2^{-mH} + c(m)) \mathbf{1}_{\{(s,r) \in [0,t]^2: s \leq r\}}$$

and therefore,

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} \sup_{(s,r) \in [0,1]^2: s \leq r} |\Psi_m(t; s, r)| = 0.$$

For any  $l = 1, \dots, [2^m t] - 1$ ,  $l' = l + 2, \dots, [2^m t]$ ,

$$\mathbf{1}_{\Delta_l^m}(s)\mathbf{1}_{\Delta_{l'}^m}(r)|A_{l'}^m(t) - A_l^m(t)| \leq C|r - s|^H.$$

Indeed, a change of variables and the assumption (ii) yield

$$\begin{aligned} \mathbf{1}_{\Delta_l^m}(s)\mathbf{1}_{\Delta_{l'}^m}(r)|A_{l'}^m(t) - A_l^m(t)| &= \mathbf{1}_{\Delta_l^m}(s)\mathbf{1}_{\Delta_{l'}^m}(r) \\ &\quad \times 2^m \left( \left| \int_{\Delta_{l'}^m} G(m)(u)du - \int_{\Delta_l^m} G(m)(u)du \right| \right) \\ &\leq C \left( \frac{l' - l}{2^m} \right)^H \leq C|r - s|^H. \end{aligned}$$

Hence

$$\sup_{m \geq 1} |\Psi_m(t; s, r)| \leq C|r - s|^H \mathbf{1}_{\{(s,r) \in [0,t]^2: s \leq r\}},$$

and consequently,

$$\sup_{m \geq 1} \sup_{t \in [0,1]} |\Psi_m(t; s, r)| \leq C|r - s|^H \mathbf{1}_{\{(s,r) \in [0,1]^2: s \leq r\}}.$$

The function  $(s, r) \mapsto \dot{h}(s)|r - s|^H \mathbf{1}_{\{(s,r) \in [0,1]^2: s \leq r\}}$  is integrable on the set  $[0, 1]^2$  with respect to the measure  $ds \times |K|(dr, s)$ . Hence,

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,t]} \int_0^1 ds \dot{h}(s) \int_0^1 K(dr, s) \Psi_m(t; s, r) = 0. \quad (80)$$

In order to complete the proof of (79), we must check that

$$\lim_{m \rightarrow \infty} \sup_{t \in [0,1]} |R_5^m(t)| = 0, \quad (81)$$

where

$$\begin{aligned} R_5^m(t) &= \sum_{l=1}^{[2^m t]-1} \int_0^t ds \dot{h}(s) \mathbf{1}_{\Delta_l^m}(s) \int_0^1 K(dr, s) \mathbf{1}_{\Delta_{[2^m t]+1}^m}(r) \\ &\quad \times \left( (G(r) - G(s)) \mathbf{1}_{\{s \leq r \leq t\}} - (A_{[2^m t]+1}^m(t) - A_l^m(t)) \right). \end{aligned}$$

For  $l = 1 \dots, [2^m t] - 1$ ,

$$\begin{aligned} \mathbf{1}_{\Delta_l^m}(s) |G(s) - A_l^m(t)| &= \mathbf{1}_{\Delta_l^m}(s) 2^m \left| \int_{\Delta_l^m} (G(m)(u) - G(s)) du \right| \\ &\leq C(2^{-mH} + c(m)) \leq C \end{aligned}$$

and

$$\mathbf{1}_{\Delta_t^m}(s) \mathbf{1}_{\Delta_{[2^m t]+1}^m}(r) \left| G(r) \mathbf{1}_{\{s \leq r \leq t\}} - A_{[2^m t]+1}^m(t) \right| \leq C.$$

Moreover, given any  $a \in ]0, H[$ ,

$$\begin{aligned} & \int_0^{t_{[2^m t]-1}^m} ds \left( \int_{\Delta_{[2^m t]+1}^m} |r-s|^{H-\frac{3}{2}} dr \right)^2 \\ & \leq C \int_0^{t_{[2^m t]-1}^m} ds (t_{[2^m t]}^m - s)^{2(1-a)(H-\frac{1}{2})} \left( (t_{[2^m t]}^m - s)^{H-\frac{1}{2}} - (t_{[2^m t]+1}^m - s)^{H-\frac{1}{2}} \right)^{2a} \\ & \leq C 2^{-2am} \int_0^{t_{[2^m t]-1}^m} ds (t_{[2^m t]}^m - s)^{2H-1-2a} \leq C 2^{-2am}. \end{aligned}$$

Hence for  $a \in ]0, H[$ ,

$$R_5^m(t) \leq C \|\dot{h}\|_2 2^{-am}.$$

This clearly implies (81) and concludes the proof of the proposition.  $\square$

The following theorem gives an integral representation of the geometric rough path  $(1, h^1, h^2, h^3)$  associated with  $h \in \mathcal{H}$ .

**Theorem 9** *Let  $h = Kh$  be an element of the reproducing kernel Hilbert space of the fractional Brownian motion with Hurst parameter  $H \in ]\frac{1}{4}, \frac{1}{2}[$ . Then for every  $s < t$ ,  $i, j, \kappa \in \{1, \dots, d\}$ ,*

$$h_{s,t}^{2,i,j} = \int_0^1 K^* \left( h_{0,\cdot}^{1,i} \mathbf{1}_{[s,t]} \right) (u) \dot{h}^j(u) du - h_{0,s}^{1,i} h_{s,t}^{1,j}, \quad (82)$$

$$h_{s,t}^{3,i,j,\kappa} = \int_0^1 K^* \left( h_{0,\cdot}^{2,i,j} \mathbf{1}_{[s,t]} \right) (u) \dot{h}^\kappa(u) du - h_{0,s}^{2,i,j} h_{s,t}^{1,\kappa} - h_{0,s}^{1,i} h_{s,t}^{2,j,\kappa}. \quad (83)$$

**Proof:** For simplicity, we shall assume  $d = 1$  and consequently, we remove the indices  $i, j, \kappa$ .

To prove (82), set  $G := h$  and  $G(m) := h(m)$ . Then owing to (38),  $G$  is  $H$ -Hölder continuous. Suppose  $[2^m s] = [2^m t]$ . Then  $|t - s| \leq 2^{-m}$  and we have

$$|h(m)(t) - h(m)(s)| \leq C 2^{-m(1-H)} |t - s| \leq C |t - s|^H.$$

Assume  $[2^m s] < [2^m t]$ . Since

$$\begin{aligned} |h(m)(t) - h(m)(s)| & \leq \left| h(m)(t) - h(m)(t_{[2^m t]}^m) \right| + \left| h(m)(t_{[2^m s]+1}^m) - h(m)(s) \right| \\ & \quad + \left| h(t_{[2^m t]}^m) - h(t_{[2^m s]+1}^m) \right|, \end{aligned}$$

the Hölder continuity of  $h$  yields

$$\sup_{m \in \mathbb{N}} |h(m)(t) - h(m)(s)| \leq C |t - s|^H.$$



Moreover,  $\sup_{t \in [0,1]} |h(m)(t) - h(t)| \leq C 2^{-mH}$ . Thus, the assumptions (i)-(iii) of Proposition 8 are satisfied, so that for every  $r \in [0, 1]$ , the sequence  $(\int_0^r h(m)(u) h(m)(du), m \geq 1)$ , converges to

$$\int_0^r h(u)h(du) = \int_0^1 K^*(h1_{]0,r[})(u) \dot{h}(u) du.$$

The construction of the geometric rough path based on  $h$  given in Proposition 4 shows that  $h_{0,r}^2 = \int_0^r h(u)h(du)$ . Then, formula (82) follows from the multiplicative properties of rough paths.

For the proof of (83), we fix  $G(\cdot) := h_{0,\cdot}^2$ , and  $G(m)(\cdot) := h(m)_{0,\cdot}^2$ . Corollary 6, Proposition 7 and the results set up in the first part of this proof show that the assumptions of Proposition 8 hold true. Therefore, for any fixed  $r \in [0, 1]$  the sequence  $(\int_0^r h(m)_{0,\cdot}^2(u)h(m)(du), m \geq 1)$  converges to  $\int_0^r h_{0,\cdot}^2(u)h(du) = \int_0^1 K^*(h_{0,\cdot}^2 1_{]0,r[})(u) \dot{h}(u) du$ . By Proposition 4, the limit must coincide with  $h_{0,r}^3$ . Then the expression (83) follows from the multiplicative property of rough paths.  $\square$

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