# Hsu-Robbins theorem for the correlated sequences

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# Hsu-Robbins and Erdös

A famous result by Hsu and Robbins (1947) says that if $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero mean and finite variance and

$$S_n := X_1 + \ldots + X_n,$$

then

$$\sum_{n \geq 1} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$.

Note that, by the law of large numbers,

$$\frac{S_n}{n} \rightarrow_{n\rightarrow\infty} 0 = \mathbf{E}(X_1)$$

so

$$P\left(|S_n| > \varepsilon n\right) = P\left(|\frac{S_n}{n}| > \varepsilon\right) \rightarrow_{n\rightarrow\infty} 0$$

for every $\varepsilon > 0$.
So, the result of Hsu-Robbins says that if the variance of $X_1$ is finite, this convergence is strong enough to ensure the summability of the series.

Later, Erdös (1949) showed that the converse implication also holds, namely if the series

$$\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right)$$

is finite for every $\varepsilon > 0$ and $X_1, X_2, \ldots$ are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$.

Since then, many authors extended this result in several directions.

Spitzer's showed that

$$\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$.

So, one introduces the factor $\frac{1}{n}$ to "help" the convergence of the series and one needs a weaker conditions on the moments

Also, Spitzer's theorem has been the object of various generalizations and variants.

One of the problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as

$$\varepsilon \longrightarrow 0$$

of the quantities

$$\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right)$$

and

$$\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right)$$

.

Obviously, these sequences goes to $\infty$ when $\varepsilon \longrightarrow 0$.

Heyde (1975) showed that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n \geq 1} P\left(|S_n| > \varepsilon n\right) = \mathbf{E} X_1^2 \tag{1}$$

whenever $\mathbf{E} X_1 = 0$ and $\mathbf{E} X_1^2 < \infty$. In the case when $X$ is attracted to a stable distribution of exponent $\alpha > 1$, Spataru proved that

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \sum_{n \geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) = \frac{\alpha}{\alpha - 1}. \tag{2}$$

It also holds for the Gaussian case : the limit is 2

# Variations of the fractional Brownian motion

Our purpose is to prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion or to moving averages sequences

Recall that the fractional Brownian motion $(B_t^H)_{t\in[0,1]}$ is a centered Gaussian process with covariance function $R^H(t,s) = \mathbf{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. It can be also defined as the unique self-similar Gaussian process with stationary increments.

Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

$$V_n = \sum_{k=0}^{n-1} H_q\left(n^H\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)\right) =_{(d)} \sum_{k=0}^{n-1} H_q\left(B_{k+1} - B_k\right) \tag{3}$$

where $B$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (in the sequel we will omit the superscript $H$ for $B$) and $H_q$ is the Hermite polynomial of degree $q \geq 1$ given by $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$.

If $q = 1$ we have $X_k = n^H \left( B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right)$ and

$$\mathbf{E} X_k X_l \neq 0$$

(unless $H = \frac{1}{2}$.)
If $q = 2$ then

$$X_k = H_2 \left( n^H \left( B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) \right)$$

$$= \left( n^H \left( B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) \right)^2 - 1.$$

In our case the variables are correlated. Indeed, for any $k, l \geq 1$ we have

$$\mathbf{E}\left(H_q(B_{k+1} - B_k)H_q(B_{l+1} - B_l)\right) = \frac{1}{(q!)}\rho_H(k-l)^q$$

where the correlation function is

$$\rho_H(k) = \frac{1}{2}\left((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right)$$

which is not equal to zero unless $H = \frac{1}{2}$ (which is the case of the standard Brownian motion).

# The convergence of the sequence $V_n$.

Let $q \geq 2$ an integer and let $(B_t)_{t\geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on $q$ and $H$ we have

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\frac{V_n}{c_{1,q,H}\sqrt{n}} \to_{n\to\infty} N(0,1) \tag{4}$$

ii. If $1 - \frac{1}{2q} < H < 1$ then

$$\frac{V_n}{c_{2,q,H}n^{1-q(1-H)}} \to_{n\to\infty} Z \tag{5}$$

where $Z$ is a Hermite random variable (an iterated stochastic integral)

Example : for $q = 2$ we have the quadratic variations of the fBm which converge as follows :
if

$$H < \frac{3}{4}$$

these variations converge (after normalization) to the normal law
and
if

$$H > \frac{3}{4}$$

these variations converges (after normalization) to a non-Gaussian law (double stochastic integral, Rosenblatt)

Our purpose : prove precise asymptotics in Hsu-Robbins theorem for $V_n$, that is look to the quantities

$$\sum_{n\geq 1} P\left(V_n > \varepsilon n\right)$$

and

$$\sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n\right)$$

-no problems related to the existence of moments
-for every $\varepsilon > 0$ the above series are convergent
- we will use chaos expansion and Malliavin calculus (the so -called Stein method)

# Multiple Wiener-Itô integrals

Let $(W_t)_{t\in[0,1]}$ a standard Wiener process.

If $f \in L^2([0,1]^n)$ we define **the multiple Wiener integral of $f$ with respect to $W$**

Let $f$ be a step function ($f \in \mathcal{S}$), that means

$$f = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} 1_{A_{i_i}\times\ldots\times A_{i_n}}$$

(here $c_{i_1,\ldots,i_n} = 0$ if two indices $i_k$ and $i_l$ are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoint). We define for such a step function

$$I_n(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} W(A_{i_1})\ldots W(A_{i_n})$$

where e.g. $W([a,b]) = W_b - W_a$.

We have that

• the application $I_n$ is an isometry on $\mathcal{S}$ , i.e.

$$\mathbf{E}\left(I_n(f)I_m(g)\right) = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n$$

and

$$\mathbf{E}\left(I_n(f)I_m(g)\right) = 0 \text{ if } m \neq n$$

• the set $\mathcal{S}$ is dense in $L^2([0,1]^n)$
Therefore $I_n$ can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$.

$I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ is the symmetrization of $f$

**Remark :** $I_n$ can be viewed as an iterated stochastic Itô integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}$$

# Hermite random variable

The Hermite random variable of order $q \geq 1$ that appears as limit in the above theorem is defined as

$$Z = d(q,H)I_q(L) \tag{6}$$

where the kernel $L \in L^2([0,1]^q)$ is given by

$$L(y_1,\ldots,y_q) = \int_{y_1 \vee \ldots \vee y_q}^{1} \partial_1 K^H(u,y_1)\ldots\partial_1 K^H(u,y_q)du.$$

The constant $d(q,H)$ is a positive normalizing constant that guarantees that $\mathbf{E}Z^2 = 1$ and $K^H$ is the standard kernel of the fractional Brownian motion. We will not need the explicit expression of this kernel. Note that the case $q = 1$ corresponds to the fractional Brownian motion and the case $q = 2$ corresponds to the Rosenblatt process.

Let us denote, for every $\varepsilon > 0$,

$$f_1(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) \qquad (7)$$

where

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}$$

while if $1-\frac{1}{2q} < H < 1$, we are interested in

$$f_2(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \tag{8}$$

where

$$Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}}$$

It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ because the $L^2$ norm of the sequence $V_n$ is in this case of order $n^{1-q(1-H)}$.

We are interested to study the behavior of $f_i(\varepsilon)$ $(i = 1, 2)$ as $\varepsilon \to 0$.

For a given random variable $X$, we set
$\Phi_X(z) = 1 - P(X < z) + P(X < -z)$.

The first lemma gives the asymptotics of the functions $f_i(\epsilon)$ as $\varepsilon \to 0$ when $Z_n^{(i)}$ are replaced by their limits.
Consider $c > 0$.

i. Let $Z^{(1)}$ be a standard normal random variable. Then as

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) \to_{\varepsilon\to 0} 2.$$

ii. Let $Z^{(2)}$ be a Hermite random variable or order $q$ given by (6). Then, for any integer $q \geq 1$

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \to_{\varepsilon\to 0} \frac{1}{1-q(1-H)}.$$

Let $q \geq 2$ and $c > 0$.
i. If $H < 1 - \frac{1}{2q}$, let $Z^{(1)}$ be standard normal random variable. Then it holds

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right]$$

$$\to_{\varepsilon\to 0} 0.$$

ii. Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right]$$

$$\rightarrow_{\varepsilon\rightarrow 0} 0.$$

# Idea of the proof

For the part i), it is based on Stein's method and Malliavin calculus ($F$ is arbitrary, $Z \sim N(0,1)$) (Nourdin -Peccati)

$$\sup_{z\in\mathbb{R}} |P(F<z) - P(Z<z)| = \sup_{z\in\mathbb{R}} \left| \mathbf{E}\left( f_z'(F) - F f_z(F) \right) \right|$$

where $f_z$ is the solution of the Stein's equation

$$1_{(-\infty,z)}(x) - P(Z<z) = f'(x) - xf(x), \qquad x \in \mathbb{R}.$$

Since

$$\mathbf{E}Ff(F) = \mathbf{E}\delta D(-L)^{-1}Ff(F) = \mathbf{E}f'(F)\langle D(-L)^{-1}F, DF\rangle$$

we obtain

$$\sup_{z\in\mathbb{R}} |P(F<z) - P(Z<z)| \leq \left(\mathbf{E}(1-\langle DF, D(-L)^{-1}F\rangle)^2\right)^{\frac{1}{2}}$$

$D$ is the Malliavin derivative, $L$ the Ornstein-Uhlenbeck operator

To get a feeling

$$D_s W_t = 1_{[0,t]}(s)$$

$$D_s W_t^n = n W_t^{n-1} D_s W_t$$

$$D_s I_n(f) = I_{n-1}(f_n(\cdot, s))$$

$$(-L)^{-1} I_n(f) = \frac{1}{n} I_n(f)$$

It follows that

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)}>x\right)-P\left(Z^{(1)}>x\right)\right|$$
$$\leq c\begin{cases}\frac{1}{\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ n^{H-1}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ n^{qH-q+\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2},1-\frac{1}{2q}).\end{cases}$$

and this implies that

$$\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)}>x\right)-P\left(Z^{(1)}>x\right)\right|$$
$$\leq c\begin{cases}\sum_{n\geq 1}\frac{1}{n\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ \sum_{n\geq 1}n^{H-2}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ \sum_{n\geq 1}n^{qH-q-\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2},1-\frac{1}{2q}).\end{cases}$$

and the last sums are finite (for the last one we use $H<1-\frac{1}{2q}$).

We state now the Spitzer's theorem for the variations of the fractional Brownian motion. Let $f_1, f_2$ be given by

$$f_1(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) \qquad (9)$$

and

$$f_2(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \qquad (10)$$

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\lim_{\varepsilon \to 0} \frac{1}{\log(c_{1,H,q}^{-1}\varepsilon)} f_1(\varepsilon) = 2.$$

ii. If $1 > H > 1 - \frac{1}{2q}$ then

$$\lim_{\varepsilon \to 0} \frac{1}{\log(c_{2,H,q}^{-1}\varepsilon)} f_2(\varepsilon) = \frac{1}{1 - q(1-H)}.$$

for every $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n\right) \tag{11}$$

if $H < 1 - \frac{1}{2q}$ and by

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \tag{12}$$

if $H > 1 - \frac{1}{2q}$. and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \to 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right)$$

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)$$

We decompose it as : for $H < 1 - \frac{1}{2q}$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) + \sum_{n\geq 1}\left[P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right)\right].$$

and for $H > 1 - \frac{1}{2q}$

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\right) + \sum_{n\geq 1}\left[P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)\right]$$

## Theorem

*Let $q \geq 2$. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$. Then*

*i. If $0 < H < 1 - \frac{1}{2q}$, we have*
$$(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \to_{\varepsilon\to 0} 1 = \mathbf{E}Z^{(1)}.$$

*ii. If $1 - \frac{1}{2q} < H < 1$ we have*
$$(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \to_{\varepsilon\to 0} \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

Joint work with Solesne Bourguin (Paris 1)
we will consider long memory moving averages defined by

$$X_n = \sum_{i \geq 1} a_i \varepsilon_{n-i}, n \in \mathbb{Z}$$

where the innovations $\varepsilon_i$ are centered i.i.d. random variables having at least finite second moments and the moving averages $a_i$ are of the form $a_i = i^{-\beta} L(i)$ with $\beta \in (\frac{1}{2}, 1)$ and $L$ slowly varying towards infinity. The covariance function $\rho(m) = \mathbf{E}(X_0 X_m)$ behaves as $c_\beta m^{-2\beta+1}$ when $m \to \infty$ and consequently is not summable since $\beta > \frac{1}{2}$. Therefore $X_n$ is usually called long-memory or "long-range dependence" moving average.

Let $K$ be a deterministic function which has Hermite rank $q$ and satisfies $\mathbf{E}(K^2(X_n)) < \infty$ and define

$$S_N = \sum_{n=1}^{N} \left[ K(X_n) - \mathbf{E}\left( K(X_n) \right) \right].$$

Suppose that the $\alpha_i$ are regularly varying with exponent $-\beta$, $\beta \in (1/2, 1)$ (i.e. $\alpha_i = |i|^{-\beta} L(i)$ and that $L(i)$ is slowly varying at $\infty$). S Then
i. If $q < (2\beta - 1)^{-1}$, then

$$h_{k,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N \underset{N \to +\infty}{\longrightarrow} Z^{(k)} \qquad (13)$$

where $Z^{(q)}$ is a Hermite random variable of order $q$

ii. If $q > (2\beta - 1)^{-1}$, then

$$\frac{1}{\sigma_{k,\beta}\sqrt{N}} S_N \underset{N\to+\infty}{\longrightarrow} \mathcal{N}(0,1) \tag{14}$$

with $\sigma_{k,\beta}$ a positive constant.
Wu (2006), H-C Ho and T. Hsing (1997), Peligrad and Utev (1997)

Take also the innovations $\varepsilon$ to be the increments of the Wiener process

$$\varepsilon_i = W_{i+1} - W_i$$

Take $K = H_q$ the Hermite polynomials

Note that $X_n$ can also be written as

$$X_n = \sum_{i=1}^{\infty} \alpha_i \left( W_{n-i} - W_{n-i-1} \right) = \sum_{i=1}^{\infty} \alpha_i I_1 \left( \mathbf{1}_{[n-i-1,n-i]} \right)$$
$$= I_1 \left( \underbrace{\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[n-i-1,n-i]}}_{f_n} \right) = I_1 \left( f_n \right). \qquad (15)$$

As $K = H_q$, $S_N$ can be represented as

$$S_N = \sum_{n=1}^{N} [H_q(I_1(f_n)) - \mathbf{E}\left(H_q(I_1(f_n))\right)] = \frac{1}{q!}\sum_{n=1}^{N}\left[I_q(f_n^{\otimes q}) - \mathbf{E}\left(I_q(f_n^{\otimes q})\right.\right.$$
$$= \frac{1}{q!}\sum_{n=1}^{N} I_q(f_n^{\otimes q}) = \frac{1}{q!} I_q(\sum_{n=1}^{N} f_n^{\otimes q}).$$

In order to apply the same techniques, we need the speed of convergence of $Z_N = cS_N/\sqrt{N}$ to the normal law, that means, we need to bound

$$\sup_{z\in\mathbb{R}} \left|\mathbf{P}(Z_N \leq z) - \mathbf{P}(Z \leq z)\right|$$

we will evaluate the quantity

$$\mathbf{E}\left(\left(1-q^{-1}\|DZ_N\|^2_{\mathcal{H}}\right)^2\right).$$

(this is the bound obtained via Malliavin calculus).
We have

$$D_tZ_N = D_t\left(\frac{1}{\sigma\sqrt{N}}\sum_{n=1}^{N} I_q\left(f_n^{\otimes q}\right)\right) = \frac{q}{\sigma\sqrt{N}}\sum_{n=1}^{N} I_{q-1}\left(f_n^{\otimes q-1}\right)f_n(t)$$

and

$$\|DZ_N\|^2_{\mathcal{H}} = \frac{q^2}{\sigma^2 N}\sum_{k,l=1}^{N} I_{q-1}\left(f_k^{\otimes q-1}\right) I_{q-1}\left(f_l^{\otimes q-1}\right)\langle f_k, f_l\rangle_{\mathcal{H}} \quad (16)$$

The multiplication formula between multiple stochastic integrals gives us that

$$I_{q-1}\left(f_k^{\otimes q-1}\right) I_{q-1}\left(f_l^{\otimes q-1}\right)$$
$$= \sum_{r=0}^{q-1} r! \left( \begin{array}{c} q-1 \\ r \end{array} \right)^2 I_{2q-2-2r}\left(f_k^{\otimes q-1-r} \tilde{\otimes} f_l^{\otimes q-1-r}\right) \langle f_k, f_l \rangle_{\mathcal{H}}^r .$$

By replacing in (16), we obtain

$$\|DZ_N\|_{\mathcal{H}}^2$$
$$= \frac{q^2}{\sigma^2 N} \sum_{r=0}^{q-1} r! \left( \begin{array}{c} q-1 \\ r \end{array} \right)^2 \sum_{k,l=1}^{N} I_{2q-2-2r}\left(f_k^{\otimes q-1-r} \tilde{\otimes} f_l^{\otimes q-1-r}\right) \langle f_k, f_l \rangle_{\mathcal{H}}^{r}$$

### Theorem

*Under the condition $q > (2\beta - 1)^{-1}$, $Z_N$ converges in law towards $Z \sim \mathcal{N}(0,1)$. Moreover, there exists a constant $C_\beta$, depending uniquely on $\beta$, such that, for any $N \geq 1$,*

$$\sup_{z\in\mathbb{R}} |\mathbf{P}(Z_N \leq z) - \mathbf{P}(Z \leq z)| \leq C_\beta \begin{cases} N^{\frac{q}{2}+\frac{1}{2}-q\beta} & \text{if } \beta \in \left(\frac{1}{2}, \frac{q}{2q-2}\right] \\ N^{\frac{1}{2}-\beta} & \text{if } \beta \in \left[\frac{q}{2q-2}, 1\right) \end{cases}$$

$$f_1(\varepsilon) = \sum_{N\geq 1} \frac{1}{N} P\left(|S_N| > \varepsilon N\right).$$

when $q > \frac{1}{2\beta - 1}$

$$\begin{aligned}
f_1(\varepsilon) &= \sum_{N\geq 1} \frac{1}{N} P\left(\sigma^{-1}\frac{1}{\sqrt{N}}|S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\
&= \sum_{N\geq 1} \frac{1}{N} P\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\
&\quad + \sum_{N\geq 1} \frac{1}{N}\left[P\left(\sigma^{-1}\frac{1}{\sqrt{N}}|S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) - P\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right)\right]
\end{aligned}$$

where $Z$ denotes a standard normal random variable.

## Proposition

*When* $q > \frac{1}{2\beta-1}$,

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_1(\varepsilon) = 2$$

*and when* $q < \frac{1}{2\beta-1}$ *then*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_2(\varepsilon) = \frac{1}{1+\frac{q}{2}-\beta q}.$$

It is also possible to give Hsu-Robbins type results, meaning to find the asymptotic behavior as $\varepsilon \to 0$ of

$$g_1(\varepsilon) = \sum_{N\geq 1} P\left(|S_N| > \varepsilon N\right)$$

when $q > \frac{1}{2\beta-1}$