PROBABILITY THEORY OF $\{n\alpha\}$

Gauss, January 30, 1812 Letter to Laplace, asking for error term in

$$\lim_{n\to\infty} \lambda\{\omega \in (0,1) : \tau^{(n)}(\omega) \le z\} = \frac{\log(1+z)}{\log 2}, \qquad \tau(\omega) = \{1/\omega\}$$

Kuzmin 1928: $r_n = O(e^{-c\sqrt{n}})$, Lévy 1929: $r_n = O(e^{-cn})$

Khinchin 1923: Small denominator problem For almost all α

$$\left|\alpha - \frac{p}{q}\right| < \frac{f(q)}{q^2} \qquad \text{i.o.} \quad \text{iff} \quad \sum \frac{f(k)}{k} = \infty$$

Erdős-Kac 1939 $\qquad \omega(n) =\#$ of prime divisors of n

$$\frac{1}{N}\#\left\{n \le N : \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x\right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

Salem-Zygmund 1947 $\qquad n_{k+1}/n_k \geq q > 1$

$$N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \longrightarrow_d N(0, 1/2)$$

Erdős-Gál 1955: corresponding LIL

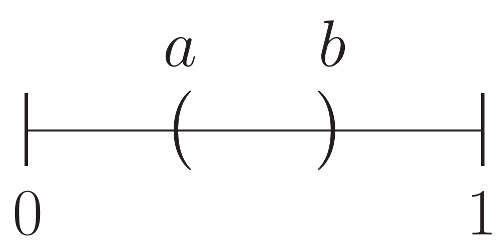
Classical theory of $\{n\alpha\}$ "Irrational rotations"

Kronecker 1876 $\qquad \{n\alpha\}$ is dense if α is irrational

Sierpinski, Bohl, Weyl 1910 $\qquad$ Uniformly distributed mod 1

$(x_n) \subset (0, 1)$ is UD if

$$\frac{1}{N} \# \big\{ k \leq N : \ x_k \in (a, b) \big\} \longrightarrow b - a$$



Discrepancy (H. Weyl 1916)

$$D_N = \sup_{0 \le a < b \le 1} \left| \frac{N(a,b)}{N} - (b-a) \right|$$

$\swarrow$

$\#$ of terms of $x_1, \ldots, x_N$ in (a,b)

(x_n) is UD $\iff D_N \longrightarrow 0$

Hardy and Littlewood 1915 Lattice points in convex domains

$$\sum_{k=1}^{N} \{kx\} = N/2 + O((\log N)^{1+\varepsilon}) \quad \text{a.s.}$$

Khinchin 1924

$$D_N(\{kx\}) = O\left(\frac{(\log N)^{1+\varepsilon}}{N} \right) \qquad \text{for } \varepsilon > 0$$

Kesten 1964

$$D_N(\{kx\}) \sim \frac{2}{\pi^2}\frac{\log N \log\log N}{N} \qquad \text{in measure.}$$

Khinchin conjecture (1924) f 1-periodic, $\int_0^1 f(x)dx = 0$

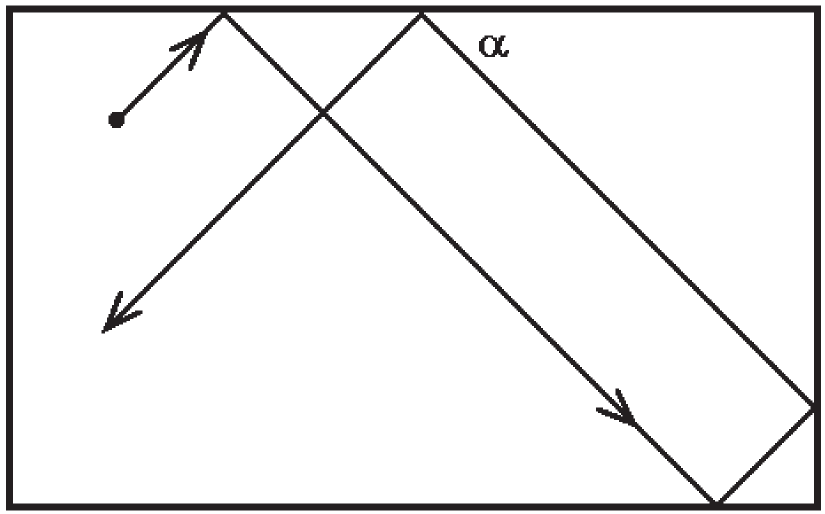
$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(kx) = 0 \qquad \text{a.e.}$$

False! (Marstrand 1969)

No satisfactory convergence theory for $\sum c_k f(kx)$

Carleson (1966) $f(x) = \sin x$,
Gaposhkin (1968) $f \in \mathrm{Lip}\,\alpha, \alpha > 1/2$
B. (1997) False if $f \in \mathrm{Lip}\,(1/2)$

Super-uniformity of the billiard path



Beck (2010) $A \subset [0,1]^2$ fix, speed $=1$

For $(1-\varepsilon)$-almost all starting positions

$$|\int_0^T I_A(X(t))dt - T\mu(A)| \le c_\varepsilon \sqrt{\log T}, \qquad T \ge T_0$$

Lacunary sequences

f 1-periodic, $\int_0^1 f(x)dx = 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(2^k x) = 0 \qquad \text{a.e.}$$

Kac (1946) Under smoothness conditions

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x) \longrightarrow_d N(0, \sigma^2)$$

where

$$\sigma^2 = \int_0^1 f^2(x)dx + 2\sum_{k=1}^{\infty} \int_0^1 f(x)f(2^k x)dx$$

Erdős-Fortet (1949) Fails for $f((2^k - 1)x)$!

Philipp-B. (1979)

$$\sum_{k=1}^{N} f(n_k x) = W(cN) + O(N^{1/2-\lambda}) \qquad \text{a.s.}$$

if n_{k+1}/n_k are integers or $n_{k+1}/n_k \to \alpha$ transcendent

Philipp (1972)

$$\limsup_{N\to\infty} \sqrt{\frac{N}{\log\log N}} D_N(\{2^k x\}) < \infty \qquad \text{a.s.}$$

The value of the limsup remained open

Fukuyama (2008)

limsup=$\sqrt{42}/9$ and the limsup is permutation-dependent

Kolmogorov (1927): There exists an $f \in L^2(0, 2\pi)$ whose Fourier series diverges a.e. after a suitable permutation.

No convergence criteria, no characterization of limits are known.

Theorem (Fukuyama) For $\{a^k x\}$ the limsup Σ_a equals

$$
\begin{aligned}
\Sigma_a &= \sqrt{42}/9 && \text{if } a = 2 \\
\Sigma_a &= \frac{\sqrt{(a+1)a(a-2)}}{2\sqrt{(a-1)^3}} && \text{if } a \ge 4 \text{ is an even integer,} \\
\Sigma_a &= \frac{\sqrt{a+1}}{2\sqrt{a-1}} && \text{if } a \ge 3 \text{ is an odd integer} \\
\Sigma_a &= 1/2 && \text{if } a^r \text{ is irrational for } r = 1, 2, \ldots
\end{aligned}
$$

For $\{(2^k - 1)x\}$, the limsup is not a constant!
But the limsup is constant if we consider only intervals $[0, a)$

Why does the behavior depend on n_k?

$$\int_a^b \left(\sum_{k=1}^{N} f(n_k x) \right)^2 dx = ?$$

Let $f(x) = a_1 \cos x + a_2 \cos 2x + \dots$

$$a_1 \sum_{k=1}^{N} \cos n_k x + a_2 \sum_{k=1}^{N} \cos 2n_k x + a_3 \sum_{k=1}^{N} \cos 3n_k x + \dots$$

$$G = \{n_k\} \cup \{2n_k\} \cup \{3n_k\} \cup \cdots$$

Landau's formula

$$I = \sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)}{[n_k, n_\ell]} \qquad \text{if } f(x) = \{x\} - 1/2$$

Harmonic case $f(x) = \sin x$

(a) $n_{k+1}/n_k \geq q > 1$ Salem-Zygmund (1947)

$$\frac{1}{A_N} \sum_{k=1}^{N} a_k \sin n_k x \longrightarrow_d N(0, 1/2)$$

provided

$$\max_{1 \leq k \leq N} |a_k| = o(A_N), \qquad A_N = \left(\sum_{k=1}^{N} a_k^2 \right)^{1/2}$$

(b) $n_k \gg e^{k^\alpha}$, $1/2 < \alpha \leq 1$ Takahashi (1965)

$$\text{CLT} \iff \max_{1 \leq k \leq N} |a_k| = o(A_N / N^{1-\alpha}) \quad \text{sharp!}$$

The trigonometric functions are "inflated"

What happens if $n_k \sim e^{\sqrt{k}}$?

(a) $n_k \sim e^{\sqrt{k}(\log\log k)^\beta}$, $\beta > 5/2$: CLT+LIL+ASIP

(b) $n_k \sim e^{\sqrt{k}(\log\log k)^\beta}$, $3/2 < \beta < 5/2$: CLT+LIL+KFT

$$S_N > \sqrt{N}\varphi_N \text{ i.o.} \Longleftrightarrow \int_1^\infty t^{-1}\varphi(t)e^{-\frac{\varphi(t)^2}{2}}\,dt = \infty$$

(c) $n_k \sim e^{\sqrt{k}(\log\log k)^\beta}$, $1/2 < \beta \le 3/2$: CLT+LIL+partly KFT

$$S_N \le (2N\log\log N)^{1/2} \quad \text{a.s. for } N \ge N_0$$

(d) $n_k \sim e^{\sqrt{k}(\log\log k)^{1/2}}$: CLT holds, but LIL not

(e) $n_k \sim e^{\sqrt{k}}$: CLT breaks down, Cauchy limit distribution

$n_k = e^{\sqrt{k}}$? $\qquad$ $n_k = e^{c\sqrt{k}}$ Kaufman (1980)

Crucial property is Diophantine behavior of (n_k).

An "arithmetic" probability theory

C. Aistleitner, I. Berkes, R. Tichy (2008-2010)

Theorem 1. *$f(n_k x)$ satisfies the CLT if and only if $L(N,a,b,c) = o(N)$ where*

$$L(N,a,b,c) = \#\{1 \le k, l \le N;\ an_k + bn_l = c\}$$

Theorem 2. If $L(N,a,b,c) = o(N/\log\log N)$, then

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e.},$$

and this condition is sharp.

$\sum_{k=1}^N a_k X_k$ CLT $a_N = o(A_N)$ LIL $a_N = o(A_N/(\log\log A_N)^{1/2})$

Theorem 3. *$f(n_k x)$ satisfies the permutation-invariant CLT iff* $L(N, a, b, c) = O(1)$.

Theorem 4. *For $n_{k+1}/n_k \geq q > 1$, $\sin n_k x$ satisfies the CLT and LIL in permutation-invariant form. For subexponential (n_k) this fails.*

Due to interlaced mixing condition.
For subexponential (n_k) we need infinite order Diophantine equations

Permutation of trigonometric series

Kolmogorov (1927): There exists an $f \in L^2(0, 2\pi)$ whose Fourier series diverges a.e. after a suitable permutation.

No convergence criteria, no characterization of limits are known.
Probability theory offers a hope to clear this up.