Truncated moments of perpetuities and a central limit theorem for GARCH(1,1) processes

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is almost surely convergent, then the distribution of U_∞ satisfies the equation

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where U and (A, B) are independent.
Moreover, if $E \log |B| < 0$, then U_∞ exists and for arbitrary U_0 the stochastic recursion

$$U_{n+1} = A_{n+1} + B_{n+1} U_n,$$

defines a sequence convergent in distribution to U_∞.

Example: squares of ARCH(1) processes

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$$X_{n+1} = \sqrt{\beta + \lambda X_n^2} Z_{n+1},$$

where $\beta, \lambda > 0$ and $\{Z_n\}$ is an i.i.d. sequence independent of X_0. We always assume that $EZ_n = 0$ i $EZ_n^2 = 1$.

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- Naturally arising sequences with "heavy tails".

Stationarity of ARCH(1)

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- If $\beta > 0$ and $\lambda \in (0, 2e^{\gamma})$, then $\{X_n\}$ is a strictly stationary sequence if and only if

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- The sequence $\{X_k^2\}$ satisfies the equation of stochastic recursion:

$$X_{k+1}^2 = \beta Z_{k+1}^2 + (\lambda Z_{k+1}^2) X_k^2 = A_{k+1} + B_{k+1} X_k^2,$$

and this is the key argument!

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$$P(X_0 > x) \sim \frac{C_{\beta,\lambda}}{2}\, x^{-2\kappa}, \text{ where}$$

$$C_{\beta,\lambda} = \frac{E\Big[\big(\beta + \lambda X_0^2\big)^{\kappa} - \big(\lambda X_0^2\big)^{\kappa}\Big]}{\kappa \lambda^{\kappa} E\Big[(\lambda Z_1^2)^{\kappa} \ln(\lambda Z_1^2)\Big]} \in (0, +\infty).$$

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- This result essentially belongs to H. Kesten (1973)!
- A complete proof, one-dimensional and using ideas of Grinkevičius (1975), belongs to C. Goldie (1991).

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$$\begin{aligned} X_n &= \sigma_n Z_n, \\ \sigma_n^2 &= \beta + \lambda X_{n-1}^2 + \delta \sigma_{n-1}^2 \,, \end{aligned}$$

where $\beta, \lambda, \delta \geqslant 0$, $\{Z_n\}$ is an i.i.d. sequence satisfying $EZ_n = 0$, $EZ_n^2 = 1$, and X_0 is independent of $\{Z_n\}$.

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  where $\beta, \lambda, \delta \geqslant 0$, $\{Z_n\}$ is an i.i.d. sequence satisfying $EZ_n = 0$, $EZ_n^2 = 1$, and X_0 is independent of $\{Z_n\}$.

- According to the relation

$$\sigma_n^2 = \beta + (\lambda Z_{n-1}^2 + \delta)\sigma_{n-1}^2 ,$$

  many of properties of GARCH(1,1) processes may be deduced from the corresponding properties of stochastic recursions.

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- If $\lambda + \delta = 1$ and $\{\sigma_n^2\}$ is stationary, then $E\sigma_n^2 = +\infty$.

Recent results for stochastic recursions

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Y. Guivarc'h, Heavy tail properties of stationary solutions of multidimensional stochastic recursions, *in:* D. Denteneer, F. den Hollander and E. Verbitskiy (Eds.), **Dynamics & Stochastics: Festschrift in honor of M. S. Keane**, *IMS Lecture Notes–Monograph Series*, **48 (2006)**, 85–99.

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Y. Guivarc'h and E. Le Page, On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks. *Ergodic Theory. Dynam. Systems*, **28 (2008)**, 423–446.

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K. Bartkiewicz, A. Jakubowski, T. Mikosch and O. Wintenberger, Stable limits for sums of dependent infinite variance random variables, *Probab. Theory Related Fields*, DOI 10.1007/s00440-010-0276-9.

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Consider the ARCH(1) recurrence with $\beta = 1$, $\lambda = 1$ and

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has the stationary distribution for squares of the corresponding ARCH(1) process. But there is no $C > 0$ such that

$$P(U_\infty > x) \sim Cx^{-1}$$

and so Kesten's theorem does not work in this simple example.

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$$EB^{\kappa} \ln B \in (0, +\infty].$$

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If

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If

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for some slowly varying function $\ell : \boldsymbol{R}^+ \to \boldsymbol{R}^+$, $\ell(x) \to \infty$, then

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$$C = \frac{E((A + BU)^{\kappa} - (BU)^{\kappa})}{\kappa EB^{\kappa} \ln B}.$$

But the theorem also shows that there exist solutions to the equation $U =_{\mathcal{D}} A + BU$, which admit different from the polynomial asymptotics of tail probabilities.

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$$\int_1^t E(\delta + \lambda Z^2) I\{(\delta + \lambda Z^2) > u\}\, du/u = \ell(\ln t),$$

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$$\sqrt{\frac{\ell(\ln n)}{n \ln n}}(X_1 + X_2 + \ldots + X_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \beta).$$

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then

$$\sqrt{\frac{\ell(\ln n)}{n \ln n}} (X_1 + X_2 + \ldots + X_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \beta).$$

Remark: if, for example, $\ell(x) = \ln x$ then we have a limit theorem with norming $\sqrt{n \ln n / \ln \ln n}$.

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- In the theorem we did not mention stationarity. The theorem holds under arbitrary initial distribution!

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$$\frac{\ell(\ln n)}{\beta n \ln n}(\sigma_0^2 + \sigma_1^2 + \ldots + \sigma_{n-1}^2) \underset{\mathcal{P}}{\longrightarrow} 1.$$

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There exists a specific result of this type.

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Then

$$\frac{Y_0^2 + Y_1^2 + \ldots + Y_{n-1}^2}{b_n^2} \xrightarrow[\mathcal{P}]{} 1.$$

CLT for GARCH(1,1) processes - conjecture

The conjectured form of the theorem

Let $\{X_k\}$ be a GARCH(1,1) process, $\beta > 0$, $\lambda + \delta = 1$. If $\{Z_k\}$ is such that

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$$\sqrt{\frac{\ell(\ln n)}{n \ln n}}(X_1 + X_2 + \ldots + X_n) \xrightarrow[\mathcal{D}]{} \mathcal{N}(0, \beta).$$