Central limit theorem for sampled sums of dependent random variables

Clémentine PRIEUR

INSA Toulouse, Institut Mathématique de Toulouse
Equipe de Statistique et Probabilités

LIMIT THEOREMS and APPLICATIONS
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Framework

\( \mathcal{E} = (\mathcal{E}, \mathcal{F}, \mu) \) a probability space, \( \mathcal{T} : \mathcal{E} \to \mathcal{E} \) bijective, bimeasurable, \( \mu \)-preserving.

We define \( (\zeta_i)_{i \in \mathbb{Z}} = \left( \mathcal{T}_i \right)_{i \in \mathbb{Z}} \) from \( (\mathcal{E}, \mu) \) to \( \mathcal{E} \).

\( (\Omega, \mathcal{A}, \mathbb{P}) \) a probability space, \( (X_i)_{i \geq 1} \) \( \geq 1 \) \( \in \mathbb{Z} \) i.i.d. on \( (\Omega, \mathcal{A}, \mathbb{P}) \).

We define \( S_n = \sum_{i=1}^{n} X_i, n \geq 1, S_0 \equiv 0 \).

For \( f \in L^1(\mu) \) and \( \omega \in \Omega \), we are interested in the sampled ergodic sums \( n^{-1} \sum_{k=0}^{n-1} f \circ \zeta S_k(\omega) \).
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We define \(S_n = \sum_{i=1}^{n} X_i, \ n \geq 1, \ S_0 \equiv 0.\)

For \(f \in L^1(\mu)\) and \(\omega \in \Omega\), we are interested in the sampled ergodic sums
\[
\sum_{k=0}^{n-1} f \circ \zeta_{S_k(\omega)}.
\]
\((S_k)_{k \geq 0}\) is universally representative for \(L^p, p > 1\) if \(\exists \Omega_0 \subset \Omega, \mathbb{P}(\Omega_0) = 1\) / for every \(\omega \in \Omega_0\), for every dynamical system \((E, \mathcal{E}, \mu, T)\), for every \(f \in L^p, p > 1\), 
\[ \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^{S_k(\omega)} \] converges \(\mu\)-almost surely.
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\[\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^{S_k(\omega)}\] converges \(\mu\)-almost surely.

**Proposition (Lacey et al.)**

Assume \(\mathbb{E}(X_1^2) < \infty.\)

Then \((S_k)_{k \geq 0}\) is universally representative for \(L^p, p > 1\) iff the random walk is transient.
We assume now that the random walk \((S_k)_{k \geq 0}\) is transient. Then, \(\forall x \in \mathbb{Z}\), the Green function \(G(0, x) = \sum_{k=0}^{+\infty} \mathbb{P}(S_k = x)\) is finite.

Examples:
1. \(\mathbb{E}|X_1| < \infty\) and \(\mathbb{E}X_1 \neq 0\),
2. \(\mathbb{E}X_1 = 0\) and \(\forall x \in \mathbb{R}, \mathbb{P}\left(n^{-1/\alpha}S_n \leq x\right) \xrightarrow{n \to +\infty} F_\alpha(x)\), where \(F_\alpha\) is the distribution function of a stable law with index \(\alpha \in (0, 1)\).
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Weak dependence framework

On the Euclidean space $\mathbb{R}^m$, define the metric $d_1(x, y) = \sum_{i=1}^m |x_i - y_i|$.

Let $\Lambda = \bigcup_{m \in \mathbb{N}^*} \Lambda_m$ where $\Lambda_m = \{ f : \mathbb{R}^m \to \mathbb{R}, \text{Lipschitz with respect to } d_1 \}$.

If $f \in \Lambda_m$, define $\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_1(x, y)}$.

Define $\tilde{\Lambda} = \{ f \in \Lambda / \text{Lip}(f) \leq 1 \}$.
On the Euclidean space $\mathbb{R}^m$, define the metric
$$d_1(x, y) = \sum_{i=1}^{m} |x_i - y_i|.$$

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Define $\tilde{\Lambda} = \{ f \in \Lambda / \text{Lip}(f) \leq 1 \}$. 
Let $\xi \in \mathbb{R}^m$ be a square integrable r.v. on $(E, \mathcal{E}, \mu)$. For any sub $\sigma$-algebra $\mathcal{M} \subset \mathcal{E}$, define

$$\theta_2(\mathcal{M}, \xi) = \sup \left\{ \| \mathbb{E}(f(\xi)|\mathcal{M}) - \mathbb{E}(f(\xi)) \|_2, \ f \in \tilde{\Lambda} \right\}.$$
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**Définition**

$$(\xi_i)_{i \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} : \text{sequence of square integrable r.v.},$$

$$\forall \; i \in \mathbb{Z}, \; \mathcal{M}_i = \sigma (\xi_j, \; j \leq i).$$

Then, $\forall \; k \in \mathbb{N}^* \cup \{\infty\}, \forall \; n \in \mathbb{N}$, define $\theta_{k,2}(n)$ by

$$\max_{1 \leq l \leq k} \frac{1}{l} \sup \left\{ \theta_2(\mathcal{M}_p, (\xi_{j_1}, \ldots, \xi_{j_l})), \; p + n \leq j_1 < \ldots < j_l \right\}.$$ 

Also define $\theta_2(n) = \theta_{\infty,2}(n) = \sup_{k \in \mathbb{N}^*} \theta_{k,2}(n)$. 

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We consider the stationary sequence \((\xi_x)_{x \in \mathbb{Z}} = (f \circ T^x)_{x \in \mathbb{Z}}\), where \(f \in L^2(\mu)\), \(\mathbb{E}_\mu(f) = 0\) and \(T : E \to E\) bijective, bimeasurable and \(\mu\)-preserving.

Assumption DEP:
Assume that \(\theta \xi_x^2(\cdot)\) is bounded above by some non-negative function \(g(\cdot)\) such that \(x \mapsto x^{3/2}g(x)\) is non-increasing, \(\exists 0 < \varepsilon < 1\), \(\sum_{i=0}^{\infty} 2^{3i/2}g(2^i \varepsilon) < \infty\).

If \(\theta \xi_n^2 = O(n^{-a})\), DEP is satisfied as soon as \(a > 3/2\).
We consider the stationary sequence $(\xi_x)_{x \in \mathbb{Z}} = (f \circ T^x)_{x \in \mathbb{Z}}$, where $f \in L^2(\mu)$, $E_{\mu}(f) = 0$ and $T : E \to E$ bijective, bimeasurable and $\mu$-preserving.

Then $\mathcal{M}_i = \sigma(f \circ T^j, j \leq i), i \in \mathbb{Z}$.

**Assumption DEP**: Assume that $\theta_2^\xi(\cdot)$ is bounded above by some non-negative function $g(\cdot)$ such that $x \mapsto x^{3/2}g(x)$ is non-increasing,

$\exists 0 < \varepsilon < 1, \sum_{i=0}^{\infty} 2^{3i/2} g(2^{i\varepsilon}) < \infty.$

If $\theta_2^\xi(n) = O(n^{-a})$, **DEP** is satisfied as soon as $a > 3/2$. 

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Main result

For \( f \in L^2(\mu) \) such that \( \mathbb{E}_\mu(f) = 0 \), define

\[
\sigma^2(f) = 2 \sum_{x \in \mathbb{Z}} G(0, x) \mathbb{E}_\mu(f \circ T^x) - \mathbb{E}_\mu(f^2).
\]

Theorem (Guillotin-Plantard & Prieur, 07)

Assume that \((f \circ T^x)_{x \in \mathbb{Z}}\) satisfies DEP. Assume that \(\sigma^2(f)\) is finite and positive.
Then, for \(\mathbb{P}\)-almost every \(\omega \in \Omega\),

\[
\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f \circ T^{S_k(\omega)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(f)).
\]
Remarks:

1. If \((f \circ T^x)_{x \in \mathbb{Z}}\) is a sequence of martingale differences, the clt holds with \(0 < \sigma^2(f) = (2G(0,0) - 1) \mathbb{E}_\mu(f^2) < \infty\) (Guillotin-Plantard & Schneider, 03).

2. The stationarity assumption on \((f \circ T^x)_{x \in \mathbb{Z}}\) can be relaxed by a stationarity assumption of order 2.
∀ \in N, \forall x \in Z, define the local time of the random walk
\[ N_n(x) = \sum_{i=0}^{1} S_i = x. \]
For every \( \omega \in \Omega \),
\[ \sum_{k=0}^{n} f \circ T S_k(\omega) = \sum_{x \in Z} N_n(x)(\omega) f \circ T x. \]
We need a clt for triangular arrays of dependent random variables.
∀ \( n \in \mathbb{N} \), ∀ \( x \in \mathbb{Z} \), define the local time of the random walk

\[
N_n(x) = \sum_{i=0}^{n} 1_{S_i = x}.
\]

For every \( \omega \in \Omega \),

\[
\sum_{k=0}^{n} f \circ T^{S_k(\omega)} = \sum_{x \in \mathbb{Z}} N_n(x)(\omega) f \circ T^x.
\]
\[ \forall \ n \in \mathbb{N}, \ \forall \ x \in \mathbb{Z}, \ \text{define the local time of the random walk} \]

\[ N_n(x) = \sum_{i=0}^{n} \mathbf{1}_{S_i = x}. \]

For every \( \omega \in \Omega \),

\[ \sum_{k=0}^{n} f \circ T^{S_k(\omega)} = \sum_{x \in \mathbb{Z}} N_n(x)(\omega)f \circ T^{x}. \]

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clt for triangular arrays of dependent r.v.

Define $X_{n,i} = a_{n,i} \xi_i$, $n = 0, 1, \ldots$, $i = -k_n, \ldots, k_n$, where

- $(\xi_i)_{i \in \mathbb{Z}}$ is a sequence of centered, non essentially constant and square integrable real valued r.v.,

- $(k_n)_{n \geq 1}$ is a ↘ sequence of positive integers s.t. $k_n \xrightarrow{n \to +\infty} +\infty$,

- $\{a_{n,i}, -k_n \leq i \leq k_n\}$ is a triangular array of real numbers s.t. $\forall n \in \mathbb{N}, \sum_{i=-k_n}^{k_n} a_{n,i}^2 > 0$.

We are interested in the asymptotic behaviour of

$$\Sigma_n = \sum_{i=-k_n}^{k_n} X_{n,i} = \sum_{i=-k_n}^{k_n} a_{n,i} \xi_i.$$
Let $M_i = \sigma (\xi_j, j \leq i)$ and $\sigma_n^2 = \text{Var} (\Sigma_n)$.

**Theorem (Guillotin-Plantard & Prieur, 07)**

Assume that

(A1)

(i) $\liminf_{n \to +\infty} \frac{\sigma_n^2}{\sum_{i=-k_n}^{k_n} a_{n,i}^2} > 0$,

(ii) $\lim_{n \to +\infty} \sigma_n^{-1} \max_{-k_n \leq i \leq k_n} |a_{n,i}| = 0$.

(A2) $\{\xi_i\}_{i \in \mathbb{Z}}$ is an uniformly integrable family.

Assume moreover that $(\xi_i)_{i \in \mathbb{Z}}$ satisfies $\mathcal{DEP}$. Then,

$$
\frac{\sum_n}{\sigma_n} \xrightarrow{\mathcal{D}} n \to +\infty \mathcal{N}(0,1).
$$
Case of sampled sums

Define \( \sigma_n^2(f) = \mathbb{E}_\mu \left( \left| \sum_{k=0}^{n} f \circ T^S_k(\omega) \right|^2 \right) \).

Recall that \( \sigma^2(f) = 2 \sum_{x \in \mathbb{Z}} G(0, x) \mathbb{E}_\mu(f \circ T_x) - \mathbb{E}_\mu(f^2) \).

**Proposition (Guillotin-Plantard & Prieur, 07)**

If \( \sum_{x \in \mathbb{Z}} G(0, x) \mathbb{E}_\mu(f \circ T^x) < +\infty \), then

\[
\frac{\sigma_n^2(f)}{n} \xrightarrow{P-a.s.} n \to +\infty \sigma^2(f).
\]
Example:
Unsymmetric random walk on nearest neighbours with \( p > q \).

\[ \forall x \geq 0, \ G(0, x) = (p - q)^{-1}, \]
\[ \forall x \leq -1, \ G(0, x) = (p - q)^{-1} \left( \frac{p}{q} \right)^x. \]

If \( f = h - h \circ T \), we get for \( \sigma^2(f) \) :

\[ -2 \frac{p - 1}{p} \mathbb{E}_\mu(h^2) + 2 \mathbb{E}_\mu(h \ h \circ T) - 2 \frac{p - q}{pq} \sum_{x \geq 1} \left( \frac{p}{q} \right)^x \mathbb{E}_\mu(h \ h \circ T^x). \]
Define $M_n = \max_{0 \leq k \leq n} |S_k|$. 

Then

$$\sum_{k=0}^{n} f \circ T^{S_k} = \sum_{|x| \leq M_n} N_n(x)f \circ T^x.$$ 

We apply the clt to the triangular array

$$\left\{ X_{n,i} = \frac{N_n(i)}{\sqrt{n}} f \circ T^i, \quad n \in \mathbb{N}, \quad -M_n \leq i \leq M_n \right\}.$$
As $f \in L^2(\mu)$, the stationary family $\{(f \circ T^i)^2\}_{i \in \mathbb{Z}}$ is uniformly integrable.
As \( f \in \mathbb{L}^2(\mu) \), the stationary family \( \{(f \circ T^i)^2\}_{i \in \mathbb{Z}} \) is uniformly integrable.

Define the self-intersection local time \( \alpha(n, x) = \sum_{i,j=0}^{n} 1_{S_i - S_j = x} \), \( x \in \mathbb{Z} \).

**Proof of (A1) (i) :**

\[
\sum_{i=-M_n}^{M_n} a_{n,i}^2 = \frac{\alpha(n,0)}{n} \xrightarrow{\mathbb{P}-a.s.} 2G(0,0) - 1 > 0
\]

(Guillotin-Plantard & Schneider, 03).

Moreover, \( \text{Var} \left( \sum_{i=1}^{n} X_{n,i} \right) = \frac{\sigma_n^2(f)}{n} \xrightarrow{n \to +\infty} \sigma^2(f) > 0 \).
Proof of $(A_1)(ii)$:

\[ \forall \rho > 0, \]

\[ \max_{-M_n \leq i \leq M_n} |a_{n,i}| = \frac{1}{\sqrt{n}} \max_{i \in \mathbb{Z}} N_n(i) = o\left(n^{\rho - \frac{1}{2}}\right) \mathbb{P}\text{-a.s..} \]

Hence

\[ \left(\sqrt{\frac{\sigma_n^2(f)}{n}}\right)^{-1} \max_{-M_n \leq i \leq M_n} |a_{n,i}| \xrightarrow[n \to +\infty]{\mathbb{P}\text{-a.s.}} 0. \]
Proof of \((A_1)(ii)\):

\[ \forall \, \rho > 0, \]

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Hence

\[ \left(\sqrt{\frac{\sigma_n^2(f)}{n}}\right)^{-1} \max_{-M_n \leq i \leq M_n} |a_{n,i}| \xrightarrow{P-a.s. \, n \to +\infty} 0. \]

\[ \Rightarrow \text{ clt for the sampled sums under DEP}. \]
Example of sampled dynamical systems

Let $I = [0, 1]$, $T: I \mapsto I$ s.t. $\exists a$ $T$-invariant probability $\mu$ on $I$.

Define $X_i = T^i$. Then $(X_i)_{i \geq 0}$ is strictly stationary from $(I, \mu)$ to $I$.

Perron-Frobenius operator:

Let $K: L^1(I, \mu) \mapsto L^1(I, \mu)$ defined by

$$\forall h \in L^1(I, \mu), \forall k \in L^\infty(I, \mu), \quad \int_0^1 K(h)(x)k(x)d\mu(x) = \int_0^1 h(x)(k \circ T)(x)d\mu(x).$$

Define $(Y_i)_{i \geq 0}$ the stationary Markov chain with invariant distribution $\mu$ and transition kernel $K$. Then, $\forall n \geq 0$, $(X_0, X_1, \ldots, X_n) \sim (Y_n, Y_{n-1}, \ldots, Y_0)$ (Gordin, 68).
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\]

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We endow the space of $BV$-functions with the norm
\[ \| h \|_v = \| dh \| + \| h \|_{1,\mu}. \]

In the case where the spectral analysis of $K$ on $(BV, \| \cdot \|_v)$ yields the existence of a spectral gap, $\exists$ a $T$-invariant $\mu \ll \lambda$ on $I$ whose density $f_\mu$ is $BV$. 
We endow the space of $BV$-functions with the norm
$$\|h\|_v = \|dh\| + \|h\|_{1,\mu}.$$ 

In the case where the spectral analysis of $K$ on $(BV, \| \cdot \|_v)$ yields the existence of a spectral gap, $\exists ! \ T$-invariant $\mu << \lambda$ on $I$ whose density $f_\mu$ is $BV$.

Assumptions: regularity, expansivity, topologically mixing (see Collet et al, 02).

Then $\exists C > 0$, $\exists 0 < \rho < 1$ s.t. 
\[ \forall \ n \geq 0, \ \theta_2^Y(n) \leq C \ \rho^n, \ \text{Dedecker & Prieur (05)}. \]
For this example, the limit variance is

\[
\sigma^2(f) = 2 \sum_{x \in \mathbb{Z}} G(0, x) \text{Cov} (f(X_0), f(X_{|x|})) - \text{Var} (f(X_0)).
\]
We observe a stationary process \((\zeta_i)_{i \in \mathbb{N}}\) at random times \((S_n)_{n \geq 0}\), with \((S_n)_{n \geq 0}\) a non-negative, increasing and transient random walk.

Sampled empirical mean: 

\[
\hat{m}_n = \frac{1}{n} \sum_{i=1}^{n} \zeta_{S_i}.
\]

A quadratic criterion: 

\[
a(S) = \lim_{n \to +\infty} n \operatorname{Var}(\hat{m}_n).
\]

If \((\operatorname{Cov}(\zeta_1, \zeta_{n+1}))_{n \in \mathbb{N}} \in l^1\), 

\[
a(S) = \sum_{k=-\infty}^{\infty} \operatorname{Cov}(\zeta_{S_1}, \zeta_{S_k+1}) < \infty.
\]
We observe a stationary process \((\zeta_i)_{i \in \mathbb{N}}\) at random times \(S_n, \ n \geq 0\), with \((S_n)_{n \geq 0}\) a non negative, increasing and transient random walk.

Sampled empirical mean: \(\hat{m}_n = \frac{1}{n} \sum_{i=1}^{n} \zeta_{S_i}\).

A quadratic criterion: \(a(S) = \lim_{n \to +\infty} (n \text{Var}\hat{m}_n)\).

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\[
a(S) = \sum_{k=-\infty}^{+\infty} \text{Cov} \left( \zeta_{S_1}, \zeta_{S_{|k|+1}} \right) < \infty.
\]
Corollaire (Guillotin-Plantard & Prieur, 07)

Assume that \((\zeta_i)_{i \in \mathbb{N}}\) satisfies \(\text{DEP}\). Assume moreover that the random walk is transient, \(S_0 = 0\) and \((S_{n+1} - S_n)_{n \in \mathbb{N}}\) takes its values in \(\mathbb{N}^*\).

Then, for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\),

\[
\sqrt{n} \left( \hat{m}_n - m \right) \xrightarrow[n \to +\infty]{} \mathcal{N} (0, a(S)).
\]

Proof : \(f = \text{Id} - m, \sigma^2(f) = a(S)\).
• And using Lindeberg-Rio + blocks? $\theta_{k,2}(n)$?

• The case where the random walk is recurrent (Pène, 07).
Thank you for your attention

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